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9. **Difference equations**
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Economic theory consists of models designed to improve our understanding of economic phenomena. Many of these models have the following structure: each member of a set of economic agents optimizes given some constraints and, given the optimal actions, variables adjust to reach some sort of equilibrium.

Consider, for example, a model in which the agents are profit-maximizing firms. Suppose that there is a single input that costs \( w \) per unit, and that a firm transforms input into output using a (differentiable) production function \( f \) and sells the output for the price \( p \). A firm's profit when it uses the amount \( x \) of the input is then

\[
p f(x) - wx.
\]

As you know, if the optimal amount of the input is positive then it satisfies the "first-order condition"

\[
p f'(x) - w = 0.
\]

Further, under some conditions on \( f \) this equation has a single solution and this solution maximizes the firm's profit. Suppose these conditions are satisfied. For any pair \( (w, p) \), denote the solution of the equation by \( z(w, p) \). Then the condition:

\[
p f'(z(w, p)) - w = 0 \text{ for all } (w, p)
\]

defines \( z(w, p) \) implicitly as a function of \( w \) and \( p \).

What can we say about the function \( z \)? Is it increasing or decreasing in \( w \) and \( p \)? How does the firm's maximized profit change as \( w \) and \( p \) change?

If we knew the exact form of \( f \) we could answer these questions by solving for \( z(w, p) \) explicitly and using simple calculus. But in economic theory we don't generally want to assume that functions take specific forms. We want our theory to apply to a broad range of situations, and thus want to obtain results that do not depend on a specific functional form. We might assume that the function \( f \) has some "sensible" properties---for example, that it is increasing---but we would like to impose as few conditions as possible. In these circumstances, in order to answer the questions about the dependence of the firm's behavior on \( w \) and \( p \) we need to find the derivatives of the implicitly-defined function \( z \). Before we do so, we need to study the chain rule and derivatives of functions defined implicitly, the next two topics. (If we are interested only in the rate of change of the firm's maximal profit with respect to the parameters \( w \) and \( p \), not in the behavior of its optimal input choice, then the envelope theorem, studied in a later section, is useful.)
Having studied the behavior of a single firm we may wish to build a model of an economy containing many firms and consumers that determines the prices of goods. In a "competitive" model, for example, the prices are determined by the equality of demand and supply for each good---that is, by a system of equations. In many other models, an equilibrium is the solution of a system of equations. To study the properties of such an equilibrium, another mathematical technique is useful.

I illustrate this technique with an example from macroeconomic theory. A simple macroeconomic model consists of the four equations

\begin{align*}
Y &= C + I + G \\
C &= f(Y - T) \\
I &= h(r) \\
r &= m(M)
\end{align*}

where \(Y\) is national income, \(C\) is consumption, \(I\) is investment, \(T\) is total taxes, \(G\) is government spending, \(r\) is the rate of interest, and \(M\) is the money supply. We assume that \(M, T,\) and \(G\) are "parameters" determined outside the system (by a government, perhaps) and that the equilibrium values of \(Y, C, I\) and \(r\) satisfy the four equations, given \(M, T,\) and \(G\).

We would like to impose as few conditions as possible on the functions \(f, h,\) and \(m\). We certainly don't want to assume specific functional forms, and thus cannot solve for an equilibrium explicitly. In these circumstances, how can we study how the equilibrium is affected by changes in the parameters? We may use the tool of differentials, another topic in this section.

### 2.2 The chain rule

#### Single variable

You should know the very important chain rule for functions of a single variable: if \(f\) and \(g\) are differentiable functions of a single variable and the function \(F\) is defined by \(F(x) = f(g(x))\) for all \(x\), then

\[F'(x) = f'(g(x))g'(x).\]

This rule may be used to find the derivative of any "function of a function", as the following examples illustrate.

**Example**

What is the derivative of the function \(F\) defined by \(F(x) = e^{\sqrt{x}}\)? If we define the functions \(f\) and \(g\) by \(f(z) = e^z\) and \(g(x) = x^{1/2}\), then we have \(F(x) = f(g(x))\) for all \(x\). Thus using the chain rule, we have

\[F'(x) = f'(g(x))g'(x) = e^{g(x)}(1/2)x^{-1/2} = (1/2)e^{g(x)}x^{-1/2}.\]

**Example**

What is the derivative of the function \(F\) defined by \(F(x) = \log x^2\)? If we define
the functions $f$ and $g$ by $f(z) = \log z$ and $g(x) = x$, then we have $F(x) = f(g(x))$, so that by the chain rule we have $F'(x) = f'(g(x))g'(x) = (1/x')2x = 2x$.

More importantly for economic theory, the chain rule allows us to find the derivatives of expressions involving arbitrary functions of functions. Most situations in economics involve more than one variable, so we need to extend the rule to many variables.

**Two variables**

First consider the case of two variables. Suppose that $g$ and $h$ are differentiable functions of a single variable, $f$ is a differentiable function of two variables, and the function $F$ of a single variable is defined by

$$F(x) = f(g(x), h(x)) \text{ for all } x.$$ 

What is $F'(x)$ in terms of the derivatives of $f$, $g$, and $h$? The **chain rule** says that

$$F'(x) = f'(g(x), h(x))g'(x) + f'(g(x), h(x))h'(x),$$

where $f'$ is the partial derivative of $f$ with respect to its $i$th argument. (This expression is sometimes referred to as the **total derivative of $F$ with respect to $x$**.)

**An extension**

We can extend this rule. If $f$, $g$, and $h$ are differentiable functions of two variables and the function $F$ of two variables is defined by

$$F(x, y) = f(g(x, y), h(x, y)) \text{ for all } x \text{ and } y,$$

then

$$F'(x, y) = f'(g(x, y), h(x, y))g'(x, y) + f'(g(x, y), h(x, y))h'(x, y),$$

and similarly for $F'(x, y)$.

More generally, we have the following result.

**Proposition**

If $g$ is a differentiable function of $m$ variables for $j = 1, \ldots, n$, $f$ is a differentiable function of $n$ variables, and the function $F$ of $m$ variables is defined by

$$F(x_1, \ldots, x_m) = f(g(x_1, \ldots, x_m), \ldots, g(x_1, \ldots, x_m)) \text{ for all } (x_1, \ldots, x_m)$$

then

$$F'(x_1, \ldots, x_n) = \sum_{j=1}^m f'_{g_j}(g(x_1, \ldots, x_n), \ldots, g(x_1, \ldots, x_n))g'_j(x_1, \ldots, x_m),$$

where $g'_j$ is the partial derivative of $g^j$ with respect to its $j$th argument.

**Example**

Consider a profit-maximizing firm that produces a single output with a single input. Denote its (differentiable) production function by $f$, the price of the input by $w$, and the price of the output by $p$. Suppose that its profit-maximizing input when the prices are $w$ and $p$ is $z(w, p)$. Then its maximized profit is

$$\pi(w, p) = pf(z(w, p)) - wz(w, p).$$

How does this profit change if $p$ increases?

Using the chain rule we have
\[ \pi'(w, p) = f(z(w, p)) + pf'(z(w, p))z'(w, p) - wz'(w, p) \]

or
\[ \pi'(w, p) = f(z(w, p)) + z'(w, p)[pf'(z(w, p)) - w]. \]

But we know that if \( z(w, p) > 0 \) then \( pf'(z(w, p)) - w = 0 \), which is the "first-order condition" for maximal profit. Thus if \( z(w, p) > 0 \) then we have
\[ \pi'(w, p) = f(z(w, p)). \]

In words, the rate of increase in the firm's maximized profit as the price of output increases is exactly equal to its optimal output.

Example

A consumer trades in an economy in which there are \( n \) goods. She is endowed with the vector \( e \) of the goods, and faces the price vector \( p \). Her demand for any good \( i \) depends on \( p \) and the value of her endowment given \( p \), namely \( p \cdot e \) (the inner product of \( p \) and \( e \), which we may alternatively write as \( \sum_{i=1}^{n} p_i e_i \)). Suppose we specify her demand for good \( i \) by the function \( f \) of \( n + 1 \) variables with values of the form \( f(p, p \cdot e) \). What is the rate of change of her demand for good \( i \) with respect to \( p_i \)?

The function \( f \) has \( n + 1 \) arguments---the \( n \) elements \( p_1, ..., p_n \) of the vector \( p \) and the number \( p \cdot e \). The derivative of \( p \) with respect to \( p_i \) is 0 for \( j \neq i \) and 1 for \( j = i \), and the derivative of \( p \cdot e \) with respect to \( p_i \) is \( e_i \). Thus by the chain rule, the rate of change of the consumer's demand with respect to \( p_i \) is
\[ f'(p, p \cdot e) + f'_{\cdot e_i}(p, p \cdot e)e_i. \]

**Leibniz's formula**

We sometimes need to differentiate a definite integral with respect to a parameter that appears in the integrand and in the limits of the integral. Suppose that \( f \) is a differentiable function of two variables, \( a \) and \( b \) are differentiable functions of a single variable, and the function \( F \) is defined by
\[ F(t) = \int_{a(t)}^{b(t)} f(t, x) \, dx \]
for all \( t \).

What is \( F'(t) \)? By the logic of the chain rule, it is the sum of three components:

- the partial derivative of the integral with respect to its top limit, times \( b'(t) \)
- the partial derivative of the integral with respect to its bottom limit, times \( a'(t) \)
- the partial derivative of the integral with respect to \( t \), holding the limits fixed.

By the **Fundamental Theorem of Calculus**, the partial derivative of the integral with respect to its top limit is \( f(t, b(t)) \), and the partial derivative of the integral with respect to its bottom limit is \( -f(t, a(t)) \). As for the last term, you might correctly guess that it is \( \int_{a(t)}^{b(t)} \frac{d}{dt} f(t, x) \, dx \),

the integral of the partial derivative of the function. We have the following result, discovered by **Gottfried Wilhelm von Leibniz** (1646-1716).

**Proposition (Leibniz's formula)**
Let $f$ be a differentiable function of two variables, let $a$ and $b$ be differentiable functions of a single variable, and define the function $F$ by
\[ F(t) = \int_{a(t)}^{b(t)} f(t, x) \, dx \quad \text{for all } t. \]
Then
\[ F'(t) = f(t, b(t))b'(t) - f(t, a(t))a'(t) + \int_{a(t)}^{b(t)} f'(t, x) \, dx. \]

As with other expressions obtained by the chain rule, we can interpret each of its parts. If $t$ changes then the limits of the integral change, and the value of the function $f$ changes at each point $x$. The change in the integral can thus be decomposed into three parts:

- the part due to the change in $b(t)$, namely $f(t, b(t))b'(t)$
- the part due to the change in $a(t)$, namely $-f(t, a(t))a'(t)$ (if $a(t)$ increases then the integral decreases)
- the part due to the change in the values of $f(t, x)$, namely $\int_{a(t)}^{b(t)} f'(t, x) \, dx$.

**Example**

The profit of a firm is $\pi(x)$ at each time $x$ from 0 to $T$. At time $t$ the discounted value of future profit is
\[ V(t) = \int_{0}^{t} \pi(x)e^{-r(t-x)} \, dx, \]
where $r$ is the discount rate. Find $V'(t)$.

Use Leibniz's rule. Define $a(t) = t$, $b(t) = T$, and $f(t, x) = \pi(x)e^{-r(t-x)}$. Then $a'(t) = 1$, $b'(t) = 0$, and $f'(t, x) = \pi(x)re^{-r(t-x)}$. Thus
\[ V'(t) = -\pi(t)e^{-rt} + \int_{0}^{t} \pi(x)re^{-r(t-x)} \, dx = -\pi(t) + rV(t). \]
The first term reflects the fact that the future is shortened when $t$ increases and the second term reflects the fact that as time advances future profit at any given time is obtained sooner, and is thus worth more.

### 2.2 Exercises on the chain rule

1. Find the derivatives of the following functions of a single variable.
   a. $f(x) = (3x^2 - 1)$
   b. $f(x) = xe^{2x}$. (Remember that $\ln(a') = x \ln a$, so that $a' = e^{\ln a}$.)
   c. $f(x) = 2^x + x^3$.
   d. $f(x) = \ln x^2$.
   e. $f(x) = \sin bx$, where $b$ is a constant.
2. Define the function $F$ of two variables by $F(x, y) = f(g(x, y), h(x, y))$ for all $(x, y)$, where $f(s, t) = st^2$, $g(x, y) = x + y$, and $h(x, y) = xy$. Use the chain rule to find $F'(x, y)$ and $F'(y, x)$.
3. Define the function $F$ of two variables by $F(x, y) = f(g(x, y), h(k(x)))$, where $f$, $g$, $h$, and $k$ are differentiable functions. Find the partial derivative of $F$ with respect to $x$ in terms of the partial derivatives of $f$, $g$, $h$, and $k$. 

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4. Define the function $F$ of two variables by $F(p, q) = pf(p, q, m(p, q))$, where $f$ and $m$ are differentiable functions. Find an expression for the partial derivative of $F$ with respect to $p$ in terms of the partial derivatives of $f$ and $m$.

5. Define the function $F$ of two variables by $F(x, y) = h(f(x), g(x, y))$, where $f$, $g$, and $h$ are differentiable functions. Find the derivative of $F$ with respect to $x$ in terms of the partial derivatives of $f$, $g$, and $h$.

6. Define the functions $U$ and $V$ of two variables by $U(x, y) = F(f(x) + g(y))$ for all $(x, y)$, and $V(x, y) = \ln[U,(x, y)/U',(x, y)]$ for all $(x, y)$, where $f$, $g$, and $F$ are twice-differentiable functions. Find $V''(x, y)$.

7. Let

$$y = F(x_i, x_2, ..., x_n, p) - p \cdot x(p),$$

where $F$ and $x_i$ for $i = 1, ..., n$ are differentiable functions, $p$ is an $n$-vector, $x(p)$ denotes the vector $(x_1, x_2, ..., x_n)$, and $p \cdot x(p)$ denotes the inner product of $p$ and $x(p)$. Find the derivative of $y$ with respect to $p_j$, given $p_j$ for $j \neq i$. (Use the notation $\partial x / \partial p_j$ for the partial derivative of $x$ with respect to $p_j$.)

8. The amount $x$ of some good demanded depends on the price $p$ of the good and the amount $a$ the producer spends on advertising: $x = f(p, a)$, with $f'(p, a) < 0$ and $f''(p, a) > 0$ for all $(p, a)$. The price depends on the weather, measured by the parameter $w$, and the tax rate $t$: $p = g(w, t)$, where $g'(w, t) > 0$ and $g''(w, t) < 0$ for all $(w, t)$. The amount of advertising depends only on $t$: $a = h(t)$, with $h'(t) > 0$. If the tax rate increases does the demand for the good necessarily increase or necessarily decrease, or neither?  

9. Let

$$H(r) = \int_{t_1}^{t_2} e^{-f(t)} \, dt,$$

where $g$ and $f$ are differentiable functions. Find $H'(r)$.

10. Let

$$H(t) = \int_{-\infty}^{\infty} g(x) e^{-\frac{(x-\delta)^2}{2\sigma^2}} \, dx,$$

where $g$ is a differentiable function and $\delta$ and $T$ are constants. Find the derivative $H'(t)$.

11. Suppose that the amount of some good demanded depends on the price $p$ of the good and the price $q$ of another good; both these prices depend on two parameters, $\alpha$ and $\beta$ (e.g. the weather, the rate of government subsidy). You observe that $\partial x / \partial \alpha > 0$, $\partial p / \partial \alpha < 0$, $\partial q / \partial \alpha > 0$, and $|\partial p / \partial \alpha| > |\partial q / \partial \alpha|$. You have a theory that $x = f(p(\alpha, \beta), q(\alpha, \beta))$, where $f'(p, q) > 0$ and $f'(p, q) > 0$ for all $(p, q)$. Are your observations consistent with your theory? Are your observations consistent with a theory that imposes the stronger restriction that $f'(p, q) > f'(p, q) > 0$ for all $(p, q)$?

12. A firm faces uncertain demand $D$ and has existing inventory $I$. The firm wants to choose its stock level $Q$ to minimize the value of the function
Given: 
\[ g(Q) = c(Q - I) + h \int_{0}^{Q} f(D) dD + p \int_{Q}^{D} f(D) dD, \]
where \( c, I, h, p, \) and \( a \) are positive constants with \( p > c \), and \( f \) is a nonnegative function that satisfies \( \int_{0}^{a} f(D) dD = 1 \) (so that it can be interpreted as a probability distribution function). (The first term represents the cost of the new stock; the second term represents the cost of overstocking; and the third term represents the cost of understocking (you miss sales, and the customers who are turned away may not come back in the future).)

a. Find \( g'(Q) \) and \( g''(Q) \) and show that \( g''(Q) > 0 \) for all \( Q \).

b. Define \( F(Q^*) = \int_{0}^{Q^*} f(D) dD \), where \( Q^* \) is the stock level that minimizes \( g(Q) \). Use the "first-order" condition \( g'(Q^*) = 0 \) to find \( F(Q^*) \) (the probability that demand \( D \) does not exceed \( Q^* \)) in terms of the parameters \( p, c, \) and \( h \). (Hint: Use the fact that \( \int_{0}^{a} f(D) dD + \int_{Q^*}^{a} f(D) dD = \int_{0}^{a} f(D) dD = 1 \).

### 2.2 Solutions to exercises on the chain rule

1. 
   a. \( f'(x) = 18x(3x^2 - 1)^2 \).
   b. \( f'(x) = e^x + x(2\ln 2)e^x \).
   c. \( f'(x) = 2\ln 2 + 2x \).
   d. \( f'(x) = 2/x \).
   e. \( f'(x) = b\cos bx \).

2. We have \( F'(x, y) = f'(g(x, y), h(x, y))g'(x, y) + f'(g(x, y), h(x, y))h'(x, y) \), \( f'(s, t) = r \), \( g'(x, y) = 1 \), \( f'(s, t) = 2st \), and \( h'(x, y) = 2xy \). Thus \( F'(x, y) = (x^2y^2 + 2x + y)^2y(2xy) = x^2y^2 + 4x^2y^2 + 4x^2y^2 = 5x^2y^2 + 4x^2y^2 \). Similarly, \( F'(x, y) = 2x^2y + 4x^2y \).

3. We have \( F'(x, y) = f'(g(x, y), h(k(x)))g'(x, y) + f'(g(x, y), h(k(x)))h'(k(x))k'(x) \).

4. We have \( F'(p, q) = f(p, q, m(p, q)) + p[f'(p, q, m(p, q))] + f'(p, q, m(p, q))m'(p, q) \).

5. We have \( U'(x, y) = F'(f(x) + g(y))f'(x) + h'(f(x) + g(y))g'(y) \).

6. We have \( V'(x, y) = F'(f(x) + g(y))f'(x) + h'(f(x) + g(y))g'(y) \).

7. The function \( F \) has 2\( n \) arguments: \( x_1(p), \ldots, x_n(p) \), and \( p, \ldots, p_n \). In applying the chain rule, we differentiate \( F \) with respect to each of these arguments, and then differentiate each argument with respect to \( p \). The derivative of each of the last \( n \) arguments except \( p \) (which is the \( (n + i) \)th argument of \( F \)) with respect to \( p \) is zero. Thus the derivative of \( F(x_1(p), \ldots, x_n(p), p) \) with respect to \( p \) is \( \sum_{i=1}^{n} F'(x_1(p), \ldots, x_i(p), p) \partial x_i/p + F'(x_1(p), \ldots, x_n(p), p) \). Now consider the last term, \(-x(p) \). We may write this term as \(-\sum_{i=1}^{n} x_i(p) \partial x_i/p \). When we differentiate the \( j \)th term in this summation, for \( j \neq i \), with respect to \( p \), we obtain \(-p \partial x_i/p \); when we differentiate the \( i \)th term with respect to \( p \), we obtain \(-x(p) - p \partial x_i/p \). In
We have \( \frac{\partial y}{\partial p} = \sum_{i=1}^n F'(x(p), \ldots, x(p), p)(\partial x/\partial p) + F'(x(p), \ldots, x(p), p) - x(p) - \sum_{i=1}^n p_i(\partial x/\partial p). \)

8. We have \( x = f(g(w, t), h(t)), \) so \( \partial x/\partial t = f'_i(g(w, t), h(t)) \cdot g'(w, t) + f'(g(w, t), h(t)) \cdot h'(t). \) Thus, given the signs of the derivatives specified in the question, demand increases as \( t \) increases.

9. \( H'(r) = e^{-\alpha r} f(g(r))g'(r) - \int_q^r te^{-\alpha t} f(t)dt. \)

10. We have

\[
H'(t) = g(t)e^{-\alpha r} - g(t - T)e^{-\alpha r} - \delta f(x)e^{-\alpha r} dx
= g(t) - g(t - T)e^{\alpha r} - \delta H(t).
\]

11. We have

\[
\frac{\partial x}{\partial \alpha} = f'_i(p(\alpha, \beta), q(\alpha, \beta)), \quad \frac{\partial p}{\partial \alpha} + f'_i(p(\alpha, \beta), q(\alpha, \beta)), \quad \frac{\partial q}{\partial \alpha},
\]

so the observations that \( \partial x/\partial \alpha > 0, \partial p/\partial \alpha < 0, \partial q/\partial \alpha > 0, \) and \( |\partial p/\partial \alpha| > |\partial q/\partial \alpha| \)
are consistent with \( f'_i(p, q) > 0 \) and \( f'_i(p, q) > 0 \) for all \( (p, q) \) \( (f'_i(p, q) \) simply has to be small enough relative to \( f'_i(p, q) \). However, the observations are not consistent with \( f'_i(p, q) > f'_i(p, q) > 0 \) for all \( (p, q) \), since in this case \( \partial x/\partial \alpha < f'_i(p, q)(\partial p/\partial \alpha + \partial q/\partial \alpha) < 0 \) since \( |\partial p/\partial \alpha| > |\partial q/\partial \alpha| \) and \( f'_i(p, q) > 0. \)

13. a. Using Leibniz' rule, we have

\[
g'(Q) = c + h\int_q^r f(D)dD - p\int_q^r f(D)dD.
\]

To find the derivative of this expression, we use Leibniz' rule on each integral separately. The derivative of \( \int_q^r f(D)dD \) is \( f(Q) \) (note that \( Q \) does not appear in the integrand, so that the last term in Leibniz' formula is 0) and the derivative of \( \int_q^r f(D)dD \) is \(-f(Q). \) Thus

\[
g''(Q) = (h + p)f(Q)
\]
and hence \( g''(Q) \leq 0 \) for all \( Q. \)

b. The first-order condition \( g'(Q^*) = 0 \) implies that

\[
c + h\int_q^r f(D)dD - p\int_q^r f(D)dD = 0.
\]

Now, \( F(Q^*) = \int_q^r f(D)dD \) and

\[
\int_q^r f(D)dD = \int_q^r f(D)dD - \int_q^r f(D)dD = 1 - F(Q^*),
\]
so that

\[
c + hF(Q^*) - p(1 - F(Q^*)) = 0,
\]
and hence \( F(Q^*) = (p-c)/(h+p). \)

2.3 Derivatives of functions defined implicitly
One parameter

The equilibrium value of a variable $x$ in some economic models is the solution of an equation of the form

$$f(x, p) = 0,$$

where $f$ is a function and $p$ is a "parameter" (a given number). In such a case, we would sometimes like to know how the equilibrium value of $x$ depends on the parameter. For example, does it increase or decrease when the value of the parameter increases?

Typically, we make assumptions about the form of the function $f$---for example, we might assume that it is increasing in $x$ and decreasing in $p$---but do not assume that it takes a specific form. Thus typically we cannot solve explicitly for $x$ as a function of $p$ (i.e. we cannot write $g(p) = \text{something}$).

We say that the equation

$$f(x, p) = 0$$

defines $x$ implicitly as a function of $p$. We may emphasize this fact by writing $f(x(p), p) = 0$ for all $p$.

Before trying to determine how a solution for $x$ depends on $p$, we should ask whether, for each value of $p$, the equation has a solution. Certainly not all such equations have solutions. The equation $x^2 + 1 = 0$, for example, has no (real) solution. Even a single linear equation may have no solution in the relevant range. If, for example, the value of $x$ is restricted to be nonnegative number (perhaps it is the quantity of a good), then for $p > 0$ the equation $x + p = 0$ has no solution.

If a single equation in a single variable has a solution, we may be able to use the Intermediate Value Theorem to show that it does. Assume that the function $f$ is continuous, the possible values of $x$ lie between $x_1$ and $x_2$, and for some value of $p$ we have $f(x_1, p) < 0$ and $f(x_2, p) > 0$, or alternatively $f(x_1, p) > 0$ and $f(x_2, p) < 0$. Then the Intermediate Value Theorem tells us that there exists a value of $x$ between $x_1$ and $x_2$ for which $f(x, p) = 0$. (Note that even if these conditions are not satisfied, the equation may have a solution.)

If we cannot appeal to the Intermediate Value Theorem (because, for example, $f$ is not continuous, or does not satisfy the appropriate conditions), we may be able to argue that a solution exists by appealing to the particular features of our equation.

Putting aside the question of whether the equation has a solution, consider the question of how a solution, if one exists, depends on the parameter $p$. Even if we cannot explicitly solve for $x$, we can find how a solution, if it exists, depends on $p$---we can find the derivative of $x$ with respect to $p$---by differentiating the equation that defines it. The principle to use is simple: if you want to find a derivative, differentiate!

Differentiating both sides of the equation $f(x(p), p) = 0$, using the chain rule, we get
\[ f'(x(p), p)x'(p) + f''(x(p), p) = 0, \]

so that
\[ x'(p) = -\frac{f'(x(p), p)}{f'(x(p), p)}. \]

Notice that even though you cannot isolate \( x \) in the original equation, after differentiating the equation you can isolate the derivative of \( x \), which is what you want.

This calculation tells you, for example, that if \( f \) is an increasing function of both its arguments \( (f'(x, p) > 0 \text{ and } f'(x, p) > 0 \text{ for all } (x, p)) \), then \( x \) is a decreasing function of \( p \).

**Application: slopes of level curves**

The equation \( f(x, y) = c \) of the level curve of the differentiable function \( f \) for the value \( c \) defines \( y \) implicitly as a function of \( x \): we can write \( f(x, g(x)) = c \) for all \( x \).

What is \( g'(x) \), the slope of the level curve at \( x \)? If we differentiate both sides of the identity \( f(x, g(x)) = c \) with respect to \( x \) we obtain
\[ f'(x, g(x))g'(x) = 0, \]
so that we can isolate \( g'(x) \):
\[ g'(x) = -\frac{f'(x, g(x))}{f'(x, g(x))} \]

or, in different notation,
\[ \frac{dy}{dx} = -\frac{f_1'(x, y)}{f_2'(x, y)}. \]

In summary, we have the following result.

**Proposition**

Let \( f \) be a differentiable function of two variables. The slope of the level curve of \( f \) for the value \( f(x_0, y_0) \) at the point \( (x_0, y_0) \) is
\[ \frac{-f_1'(x_0, y_0)}{f_2'(x_0, y_0)}. \]

We deduce that the equation of the tangent to the level curve at \( (x_0, y_0) \) is
\[ y - y_0 = -\frac{f_1'(x_0, y_0)}{f_2'(x_0, y_0)}(x - x_0). \]

(Remember that the equation of a line through \( (x_0, y_0) \) with slope \( m \) is given by \( y - y_0 = m(x - x_0) \).) Thus the equation of the tangent may alternatively be written as
\[ f_1'(x_0, y_0)(x - x_0) + f_2'(x_0, y_0)(y - y_0) = 0, \]

or
\[ (f_1'(x_0, y_0), f_2'(x_0, y_0)) \{ x - x_0 \} = 0. \]
The vector $(f_1'(x_0, y_0), f_2'(x_0, y_0))$ is called the **gradient vector** and is denoted $\nabla f(x_0, y_0)$.

Let $(x, y) \neq (x_0, y_0)$ be a point on the tangent at $(x_0, y_0)$. Then the vector

\[
x - x_0
\]

is parallel to the tangent. The previous displayed equation, in which the product of this vector with the gradient vector is 0, shows that the two vectors are orthogonal (the angle between them is $90^\circ$). Thus the gradient vector is orthogonal to the tangent, as illustrated in the following figure.

One can compute the **second** derivative of the level curve as well as the first derivative, by differentiating once again.

**Many parameters**

Suppose that the equilibrium value of the variable $x$ is the solution of an equation of the form

\[
f(x, p) = 0,
\]

where $p$ is a vector of parameters---$p = (p_1, ..., p_n)$, say. By differentiating the equation with respect to $p_i$, holding all the other parameters fixed, we may determine how $x$ varies with $p_i$.

Recording the dependence of $x$ on $p$ explicitly in the notation, we have

\[
f(x(p), p) = 0 \text{ for all } p.
\]

Differentiating this identity with respect to $p_i$, we have

\[
f_1'(x(p), p)x'(p) + f_{p_1}(x(p), p) = 0
\]

so that

\[
x'(p) = -\frac{f_{p_1}(x(p), p)}{f_1'(x(p), p)}.
\]
Example
Consider the competitive firm studied previously that uses a single input to produce a single output with the differentiable production function \( f \), facing the price \( w \) for the input and the price \( p \) for output. Denote by \( z(w, p) \) its profit-maximizing input for any pair \((w, p)\). We know that \( z(w, p) \) satisfies the first-order condition
\[
p f'(z(w, p)) - w = 0 \text{ for all } (w, p).
\]
How does \( z \) depend on \( w \) and \( p \)?

Differentiating with respect to \( w \) the equation that \( z(w, p) \) satisfies we get
\[
p f''(z(w, p)) z'(w, p) - 1 = 0.
\]
Thus if \( f''(z(w, p)) \neq 0 \) then
\[
z'(w, p) = \frac{1}{p f''(z(w, p))}.
\]
We know that \( f''(z(w,p)) \leq 0 \) given that \( z(w, p) \) is a maximizer, so that if \( f''(z(w, p)) \neq 0 \) we conclude that \( z'(w, p) < 0 \), which makes sense: as the input price increases, the firm’s optimal output decreases.

A similar calculation yields
\[
z'(w, p) = -\frac{f'(z(w, p))}{p f''(z(w, p))},
\]
which for the same reason is positive.

2.3 Exercises on derivatives of functions defined implicitly

1. Suppose that \( 2x^3 + 6xy + y^2 = c \) for some constant \( c \). Find \( \frac{dy}{dx} \).
2. Suppose that the functions \( f \) and \( g \) are differentiable and \( g(f(x)) = x \) for all values of \( x \). Use implicit differentiation to find an expression for the derivative \( f'(x) \) in terms of the derivative of \( g \).
3. The demand and the supply for a good both depend upon the price \( p \) of the good and the tax rate \( t \): \( D = f(p, t) \) and \( S = g(p, t) \). For any given value of \( t \), an equilibrium price is a solution of the equation \( f(p, t) = g(p, t) \). Assume that this equation defines \( p \) as a differentiable function of \( t \). Find \( \frac{dp}{dt} \) in terms of the partial derivatives of \( f \) and \( g \).
4. The demand for a good both depends upon the price \( p \) of the good and the tax rate \( t \): \( D = f(p, h(t)) \). The supply of the good depends on the price: \( S = g(p) \). For any given value of \( t \), an equilibrium price is a solution of the equation \( f(p, h(t)) = \)
g(p). Assume that this equation defines p as a differentiable function of t. Find \( \frac{\partial p}{\partial t} \) in terms of the derivatives of \( f \), \( g \), and \( h \).

5. Let \( f(x, y) = 2x^2 + xy + y^3 \).
   a. Find the equation of the tangent at \( (x, y) = (2, 0) \) to the level curve of \( f \) that passes through this point.
   b. Find the points at which the slope of the level curve for the value 8 is 0.

6. Let \( D = f(r, P) \) be the demand for an agricultural product when the price is \( P \) and the producers' total advertising expenditure is \( r \); \( f \) is decreasing in \( P \). Let \( S = g(w, P) \) be the supply, where \( w \) is an index of how favorable the weather has been; \( g \) is increasing in \( P \). Assume that \( g'(w, P) > 0 \). An equilibrium price satisfies \( f(r, P) = g(w, P) \); assume that this equation defines \( P \) implicitly as a differentiable function of \( r \) and \( w \). Find \( \frac{\partial P}{\partial w} \) and determine its sign.

7. The equilibrium value of the variable \( x \) is the solution of the equation

\[
f(x, \alpha, \beta) + g(h(x), k(\alpha)) = 0,
\]
where \( \alpha \) and \( \beta \) are parameters and \( f \), \( g \), \( h \), and \( k \) are differentiable functions. How is the equilibrium value of \( x \) affected by a change in the parameter \( \alpha \) (holding \( \beta \) constant)?

8. The equilibrium value of the variable \( x \) is the solution of the equation

\[
f(x, g(x, \alpha), \beta) + h(x, \beta) = 0,
\]
where \( \alpha \) and \( \beta \) are parameters and \( f \), \( g \), and \( h \) are differentiable functions. How is the equilibrium value of \( x \) affected by a change in the parameter \( \alpha \) (holding \( \beta \) constant)?

9. The equilibrium value of the variable \( x \) depends on the parameters \( (a_1, \ldots, a_n) \):

\[
f(a_1, \ldots, a_n, x) = 0.
\]
Find the rate of change of \( x \) with respect to \( a_i \) for any \( i = 1, \ldots, n \).

10. The value of \( y \) is determined as a function of \( t \) by the equation

\[
\int_1^t f(x, y)dx = 0,
\]
where \( f \) is a differentiable function. Find \( dy/dt \).

11. The function \( g \) is defined implicitly by the condition \( F(f(x, y), g(y)) = h(y) \). Find the derivative \( g'(y) \) in terms of the functions \( F \), \( f \), \( g \), and \( h \) and their derivatives.

12. Suppose that \( x \) is implicitly defined as a function of the parameter \( t \) by the equation \( f(x)/f'(x) - x = t \), where \( f \) is a twice-differentiable function. Find \( dx/dt \).

2.3 Solutions to exercises on derivatives of functions defined implicitly
1. \( \frac{dy}{dx} = \frac{-2x + 3y}{3x + y} \).

2. \( g'(f(x)) = 1 \), so \( f'(x) = \frac{1}{g'(f(x))} \).

3. We have \( f'(p, t) = \frac{g'(p, t) - f'(p, t)}{f'(p, t) - g'(p, t)} \), and so

\[
\frac{\partial p}{\partial t} = \frac{g'(p, t) - f'(p, t)}{f'(p, t) - g'(p, t)}.
\]

4. We have

\[
\frac{\partial p}{\partial t} = \frac{f'(p, h(t))h'(t)}{g'(p) - f'(p, h(t))}.
\]

5. a. The slope at \((x, y)\) of the level curve through \((x, y)\) is \( \frac{-f'(x, y)}{f'(x, y)} = \frac{-(4x + y)/(x + 2y)}{-(4x + y)/(x + 2y)} \). Thus the slope of the tangent at \((2, 0)\) to the level curve through \((2, 0)\) is \(-8/2 = -4\). The equation of the straight line through an arbitrary point \((x, y)\) with slope \(m\) is \(y - y_0 = m(x - x_0)\), so the equation of the tangent at \((2, 0)\) to the level curve that passes through this point is \(y = -4(x - 2)\) or \(y = -4x + 8\).

b. The slope of a level curve is 0 if and only if \(2x^2 + xy + y^2 = 0\). The point \((x, y)\) is on the level curve for the value 8 if \(2x^2 + xy + y^2 = 8\). The two equations imply that \(2x^2 - 4x^2 + 16x^2 = 8\), or \(7x^2 = 4\), or \(x = \pm\sqrt{7}/\sqrt{7}\). Thus the points at which the slope of the level curve for the value 8 is 0 are \((a, -4a)\) and \((-a, 4a)\), where \(a = 2\sqrt{7}/7\).

6. We have

\[
\frac{\partial P}{\partial w} = -\frac{g'(w, P)}{g'(w, P) - f'(r, P)} < 0.
\]

7. (the sign follows from \(f' > 0, g' > 0, \) and \(g' > 0\)).

8. Differentiating with respect to \(\alpha\) we obtain

\[
f'(x, \alpha, \beta) \frac{\partial x}{\partial \alpha} + f'(x, \alpha, \beta) + g'(h(x), k(\alpha))h'(x) \frac{\partial x}{\partial \alpha} + g'(h(x), k(\alpha))k'(\alpha) = 0,
\]

so that

\[
\frac{\partial x}{\partial \alpha} = \frac{-f'(x, \alpha, \beta) - g'(h(x), k(\alpha))k'(\alpha)}{f'(x, \alpha, \beta) + g'(h(x), k(\alpha))h'(x)}.
\]
10. Differentiating with respect to $\alpha$ we obtain
\[
\frac{\partial x}{\partial \alpha} = \frac{f'(x, g(x, \alpha, \beta))}{f'(x, g(x, \alpha, \beta)) + f'(x, g(x, \alpha, \beta))g'(x, \alpha) + h'(x, \beta)}.
\]

12. so that
\[
\frac{\partial x}{\partial \alpha} = -\frac{f'(x, g(x, \alpha, \beta))g'(x, \alpha)}{f'(x, g(x, \alpha, \beta)) + f'(x, g(x, \alpha, \beta))g'(x, \alpha) + h'(x, \beta)}.
\]

13. The equation defines $x$ implicitly as a function of $a_i$ (and all the other parameters). Differentiating it with respect to $a_i$ we get
\[
f'(a_1, \ldots, a_n, x) + \frac{f'(x, g(x, \alpha, \beta)g'(x, \alpha)}{f'(x, g(x, \alpha, \beta)) + f'(x, g(x, \alpha, \beta))g'(x, \alpha) + h'(x, \beta)} = 0,
\]

14. Differentiating with respect to $t$ we have
\[
-f(t, y) + \int f'(x, y)y'(t)dx = 0,
\]
so that
\[
y'(t) = \frac{f(t, y)}{\int f'(x, y)dx}.
\]

15. We have $F'(f(x,y),g(y))f'(x,y) + F'(f(x,y),g(y))g'(y) = h'(y)$, so that
\[
g'(y) = \frac{h'(y) - F'(f(x,y),g(y))f'(x,y)}{F'(f(x,y),g(y))}.
\]

16. Differentiate the equation defining $x$ with respect to $t$:
\[
[(f'(x))^2 - f(x)f''(x)]dx/dt - dx/dt = 1,
\]
so that
\[ \frac{dx}{dt} = -\frac{[f'(x)]'}{[f(x)f''(x)]}. \]

### 2.4 Differentials and comparative statics

#### Introduction

We may use the tool of implicit differentiation to study the dependence of a variable on parameters when the variable is defined by an equation like \( f(x, p) = 0 \), where \( x \) is a list of variables and \( p \) is a list of parameters. Many models in economic theory involve several equations simultaneously. In these cases, another (closely related) method is useful.

Suppose that we have two variables, \( x \) and \( y \), and two parameters \( p \) and \( q \), and for any values of \( p \) and \( q \) the values of the variables satisfy the two equations

\[
\begin{align*}
 f(x, y, p, q) &= 0 \\
g(x, y, p, q) &= 0.
\end{align*}
\]

These two equations implicitly define \( x \) and \( y \) as functions of \( p \) and \( q \). As in the case of a single equation, two questions arise:

- Do the equations have solutions for \( x \) and \( y \) for any given values of \( p \) and \( q \)?
- How do the solutions change as \( p \) or \( q \), or possibly both, change?

#### Existence of a solution

We have seen that even a single equation in a single variable may not have a solution, but that if it does, we may be able to use the Intermediate Value Theorem to show that it does. Generalizations of the Intermediate Value Theorem, which I do not discuss, can be helpful in showing that a collection of equations in many variables has a solution.

A useful rough guideline for a set of equations to have a unique solution is that the number of equations be equal to the number of variables. This condition definitely is neither necessary nor sufficient, however. For example, the single equation \( x^2 = -1 \) in a single variable has no solution, while the single equation \( x^2 + y^2 = 0 \) in two variables has a unique solution \((x, y) = (0, 0))\). But there is some presumption that if the condition is satisfied and the equations are all “independent” of each other, the system is likely to have a unique solution, and if it is not satisfied there is little chance that it has a unique solution.

#### Differentials

Now consider the question of how a solution, if it exists, depends on the parameters. A useful tool to address this question involves the notion of a differential.
Let $f$ be a differentiable function of a single variable. If $x$ increases by a small amount from $a$ to $a + \Delta x$, by how much does $f(x)$ increase? A function of a single variable is approximated at $a$ by its tangent at $a$. Thus if $\Delta x$ is very small then the approximate increase in $f(x)$ is

$$f'(a)\Delta x$$

(where $f'(a)$ is of course the derivative of $f$ at $a$).

For any change $dx$ in $x$ we define the differential of $f(x)$ as follows.

**Definition**

Let $f$ be a function of a single variable. For any real number $dx$, the differential of $f(x)$ is

$$f'(x)dx.$$  

By the argument above, if $dx$ is small then the differential $f'(x)dx$ is approximately equal to the change in the value of $f$ when its argument increases or decreases by $dx$ from $x$.

If $f$ is a function of two variables, it is approximated by its tangent plane: for $(x, y)$ close to $(a, b)$ the approximate increase in $f(x, y)$ when $x$ changes by $\Delta x$ and $y$ changes by $\Delta y$ is

$$f',(a, b)\Delta x + f'_y(a, b)\Delta y.$$  

For a function of many variables, the differential is defined as follows.

**Definition**

Let $f$ be a function of many variables. For any real numbers $dx_1, ..., dx_n$, the differential of $f(x_1, ..., x_n)$ is

$$f'(x_1, ..., x_n)dx_1 + ... + f'(x_1, ..., x_n)dx_n.$$  

As in the case of a function of a single variable, if $dx_i$ is small for each $i = 1, ..., n$, then the differential $f'(x_1, ..., x_n)dx_1 + ... + f'(x_1, ..., x_n)dx_n$ is approximately equal to the change in the value of $f$ when each argument $x_i$ changes by $dx_i$.  

To find a differential we may simply find the partial derivatives with respect to each variable in turn. Alternatively we can use a set of rules that are analogous to those for derivatives. Denoting the differential of the function \( f \) by \( d(f) \), we have:

\[
\begin{align*}
\frac{d(a f + bg)}{dx} &= ad f + bdg \\
\frac{d(f \cdot g)}{dx} &= gdf + f dg \\
\frac{d(f / g)}{dx} &= (gd f - f dg) / g^2
\end{align*}
\]

if \( z = g(f(x, y)) \) then \( dz = g'(f(x, y)) df \)

**Comparative statics**

Start with the simplest case: a single equation in one variable and one parameter:

\( f(x, p) = 0 \) for all \( x \),

where \( x \) is the variable and \( p \) the parameter. We have previously seen how to use implicit differentiation to find the rate of change of \( x \) with respect to \( p \). We may reach the same conclusion using differentials. The differential of the left-hand side of the equation is

\[
\frac{df}{dx} dx + \frac{df}{dp} dp.
\]

When \( p \) changes, the value of \( f(x, p) \) must remain the same for the equation \( f(x, p) = 0 \) to remain satisfied, so for small changes \( dx \) in \( x \) and \( dp \) in \( p \) we must have, approximately,

\[
f'(x, p) dx + f'(x, p) dp = 0.
\]

Rearranging this equation we have

\[
\frac{dx}{dp} = -\frac{f'(x, p)}{f'(x, p)}.
\]

The entity on the left-hand side is the quotient of the small quantities \( dx \) and \( dp \), not a derivative. However, we can in fact show that the right-hand side is the derivative of \( x \) with respect to \( p \), as we found previously.

This technique may be extended to systems of equations. Suppose, for example, that the variables, \( x \), and \( y \), satisfy the following two equations, where \( p \) and \( q \) are parameters, as in the opening section above:

\[
\begin{align*}
f(x, y, p, q) &= 0 \\
g(x, y, p, q) &= 0.
\end{align*}
\]

Assume that the functions \( f \) and \( g \) are such that the two equations define two solution functions \( x^*(p, q) \) and \( y^*(p, q) \).

That is, \( f(x^*(p, q), y^*(p, q), p, q) = 0 \) and \( g(x^*(p, q), y^*(p, q), p, q) = 0 \) for all \( p \) and all \( q \).

How do \( x^* \) and \( y^* \) depend on the parameters \( p \) and \( q \)? Assuming that the functions \( x^* \) and \( y^* \) are differentiable, we can answer this question by calculating the differentials of the functions on each side of the two equations defining them. If the changes in \( p \) and \( q \) are small, then the differentials must be equal, so that the equations defining \( x^* \) and \( y^* \) remain satisfied. That is,

\[
f_{1'} \cdot dx + f_{1'} \cdot dy + f_{2'} \cdot dp + f_{2'} \cdot dq = 0
\]
\[ g' \cdot dx + g' \cdot dy + g' \cdot dp + g' \cdot dq = 0. \]

(To make these equations easier to read, I have omitted the arguments of the partial derivatives.)

To find the changes \(dx\) and \(dy\) in \(x\) and \(y\) necessary for these equations to be satisfied we need to solve the equations for \(dx\) and \(dy\) as functions of \(dp\) and \(dq\), the changes in the parameters. (See the page on matrices and solutions of systems of simultaneous equations if you have forgotten how.) We obtain

\[
dx = \frac{-g' \cdot (f' \cdot dp + f' \cdot dq) + f' \cdot (g' \cdot dp + g' \cdot dq)}{f' \cdot g' - f' \cdot g'}
\]

and

\[
dy = \frac{g' \cdot (g' \cdot dp + g' \cdot dq) - f' \cdot (g' \cdot dp + g' \cdot dq)}{f' \cdot g' - f' \cdot g'}.
\]

Now, to determine the impact on \(x\) and \(y\) of a change in \(p\), holding \(q\) constant, we set \(dq = 0\) to get

\[
dx = \frac{(-g' \cdot f' + f' \cdot g') \cdot dp}{f' \cdot g' - f' \cdot g'}
\]

and

\[
dy = \frac{(g' \cdot g' - f' \cdot g') \cdot dp}{f' \cdot g' - f' \cdot g'}.
\]

We can alternatively write the first equation, for example, as

\[
\frac{\partial x}{\partial p} = -\frac{g' \cdot f' + f' \cdot g'}{f' \cdot g' - f' \cdot g'}.
\]

If we make some assumption about the signs of the partial derivatives of \(f\) and \(g\), this expression may allow us to determine the sign of \(\partial x/\partial p\)---that is, the direction in which the equilibrium value of \(x\) changes when the parameter \(p\) changes.

This technique allows us also to study the change in a variable when more than one parameter changes, as illustrated in the following economic example.

**Example**

Consider the macroeconomic model

\[
Y = C + I + G
\]

\[
C = f(Y - T)
\]

\[
I = h(r)
\]

\[
r = m(M)
\]

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where the variables are \( Y \) (national income), \( C \) (consumption), \( I \) (investment) and \( r \) (the rate of interest), and the parameters are \( M \) (the money supply), \( T \) (the tax burden), and \( G \) (government spending). We want to find how the variables change with the parameters.

Take differentials:

\[
\begin{align*}
 dY &= dC + dl + dG \\
 dC &= f'(Y - T)(dY - dT) \\
 dl &= h'(r)dr \\
 dr &= m'(M) dM \\
\end{align*}
\]

We need to solve for \( dY, dC, dl, \) and \( dr \) in terms of \( dM, dT, \) and \( dG. \) The system is too big to use Cramer's rule easily, but it has enough structure to proceed step-by-step.

From the last two equations we have

\[ dl = h'(r) m'(M) dM. \]

Now substitute for \( dl \) in the first equation to get

\[
 dY - dC = h'(r) m'(M) dM + dG
\]

\[ f'(Y - T) dY - dC = f'(Y - T) dT \]

You can solve this system (using Cramer's rule) for \( dY \) and \( dC. \) For example,

\[
 dY = \frac{h'(r) m'(M)}{1 - f'(Y - T)} dM - \frac{f'(Y - T)}{1 - f'(Y - T)} dT + \frac{1}{1 - f'(Y - T)} dG.
\]

Thus, for example, if \( T \) changes while \( M \) and \( G \) remain constant (so that \( dM = dG = 0 \)), then the rate of change of \( Y \) is given by

\[
 \frac{\partial Y}{\partial T} = -\frac{f'(Y - T)}{1 - f'(Y - T)}.
\]

That is, if \( 0 < f'(z) < 1 \) for all \( z \) then \( Y \) decreases as \( T \) increases. Further, we can deduce that if \( T \) and \( G \) increase by equal (small) amounts then the change in \( Y \) is

\[
 dY = \frac{1 - f'(Y - T)}{1 - f'(Y - T)} dT = dT.
\]

That is, an equal (small) increase in \( T \) and \( G \) leads to an increase in \( Y \) of the same size.

### 2.4 Exercises on differentials

1. Find the differentials of the following.
   a. \( z = xy^3 + x^4. \)
   b. \( z = a_1 x_1^2 + \ldots + a_n x_n^2 \) (where \( a_1, \ldots, a_n \) are constants).
c. \[ z = A(\alpha_1 x_1^{\rho} + ... + \alpha_n x_n^{\rho})^{1/\rho} \] (where \( A, \rho, \) and \( \alpha_1, ..., \alpha_n \) are constants). [This is a constant elasticity of substitution function.]

2. Consider the system of equations

\[
\begin{align*}
xu^3 + v &= y^2 \\
3uv - x &= 4
\end{align*}
\]

a. Take the differentials of both equations and solve for \( du \) and \( dv \) in terms of \( dx \) and \( dy \).

b. Find \( \partial u / \partial x \) and \( \partial v / \partial x \) using your result in part (a).

3. The equilibrium values of the variables \( x, y, \) and \( \lambda \) are determined by the following set of three equations:

\[
\begin{align*}
U'_1(x, y) &= \lambda p \\
U'_2(x, y) &= \lambda q \\
px + qy &= I
\end{align*}
\]

where \( p, q, \) and \( I \) are parameters and \( U \) is a twice differentiable function. Find \( \partial x / \partial p \). [Use Cramer's rule to solve the system of equations you obtain.]

4. The equilibrium values of the variables \( Y, C, \) and \( I \) are given by the solution of the three equations

\[
\begin{align*}
Y &= C + I + G \\
C &= f(Y, T, r) \\
I &= h(Y, r)
\end{align*}
\]

where \( T, G, \) and \( r \) are parameters and \( f \) and \( h \) are differentiable functions. How does \( Y \) change when \( T \) and \( G \) increase by equal amounts?

5. An industry consists of two firms. The optimal output of firm 1 depends on the output \( q_2 \) of firm 2 and a parameter \( \alpha \) of firm 1's cost function: \( q_1 = f(q_2, \alpha) \), where \( f'(q_2, \alpha) < 0 \) and \( f'(q_2, \alpha) > 0 \) for all \( q_2 \) and all \( \alpha \). The optimal output of firm 2 depends on the output \( q_1 \) of firm 1: \( q_2 = g(q_1) \), where \( g'(q_1) < 0 \). The equilibrium values of \( q_1 \) and \( q_2 \) are thus determined as the solution of the simultaneous equations

\[
\begin{align*}
q_1 &= f(q_2, \alpha) \\
q_2 &= g(q_1)
\end{align*}
\]

8. Is this information sufficient to determine whether an increase in \( \alpha \) increases or decreases the equilibrium value of \( q_1 \)?

9. The variables \( x \) and \( y \) are determined by the following pair of equations:

\[
\begin{align*}
f(x) &= g(y) \\
Ay + h(x) &= \beta
\end{align*}
\]
10. where \( f, g, \) and \( h \) are given differentiable functions, \( \beta \) is a constant, and \( A \) is a parameter. By taking differentials, find \( \partial x / \partial A \) and \( \partial y / \partial A \).

11. The optimal advertising expenditure of politician 1 depends on the spending \( s_1 \) of politician 2 and a parameter \( \alpha \): \( s_1 = f(s_2, \alpha) \), where \( 0 < f'(s_2, \alpha) < 1 \) and \( f'(s_2, \alpha) < 0 \) for all \( s_2 \) and all \( \alpha \). The optimal expenditure of politician 2 depends on the spending \( s_2 \) of politician 1 and a parameter \( \beta \): \( s_2 = g(s_1, \beta) \), where \( 0 < g'(s_1, \beta) < 1 \) and \( g'(s_1, \beta) < 0 \) for all \( s_2 \) and all \( \alpha \). The equilibrium values of \( s_1 \) and \( s_2 \) are given by the solution of the simultaneous equations

\[
\begin{align*}
  s_1 &= f(s_2, \alpha) \\
  s_2 &= g(s_1, \beta).
\end{align*}
\]

12. Does an increase in \( \alpha \) (holding \( \beta \) constant) necessarily increase or necessarily decrease the equilibrium value of \( s_1 \)?

13. The equilibrium outputs \( q_1 \) and \( q_2 \) of two firms satisfy

\[
\begin{align*}
  q_1 &= b(q_2, c_1) \\
  q_2 &= b(q_1, c_2),
\end{align*}
\]

14. where \( b \) is a differentiable function that is decreasing in each of its arguments and satisfies \( b'(q, c) > -1 \) for all \( q \) and \( c \), and \( c_1 \) and \( c_2 \) are parameters.
   a. Find the differentials of the pair of equations.
   b. Find the effect on the values of \( q_1 \) and \( q_2 \) of equal increases in \( c_1 \) and \( c_2 \)
      starting from a situation in which \( c_1 = c_2 \) and an equilibrium in which \( q_1 = q_2 \).

15. The equilibrium values of the variables \( Y \) and \( r \) are given by the solution of the two equations

\[
\begin{align*}
  I(r) &= S(Y) \\
  aY + L(r) &= M
\end{align*}
\]

16. where \( a > 0 \) and \( M \) are parameters, \( I \) is an increasing differentiable function, and \( S \) and \( L \) are decreasing differentiable functions. How do \( Y \) and \( r \) change when \( M \) increases (holding \( a \) constant)?

17. Consider a market containing two goods. Denote the prices of these goods by \( p \) and \( q \). Suppose that the demand for each good depends on \( p, q, \) and the amount of advertising expenditure \( a \) on good 1, and that the supply of each good depends only on the price of that good. Denoting the demand functions by \( x \) and \( y \) and the supply functions by \( s \) and \( t \), for any given value of \( a \) a market equilibrium is a pair \((p, q)\) or prices such that

\[
\begin{align*}
  x(p, q, a) &= s(p) \\
  y(p, q, a) &= t(q).
\end{align*}
\]

18. How does the equilibrium price \( p \) of good 1 change as \( a \) changes?
19. Assume that $x' \, p < 0$, $x' \, q > 0$, $x' \, a > 0$, $s' \, q > 0$, $y' \, q < 0$, $y' \, a < 0$, and $t' > 0$. (What are the economic interpretations of these assumptions?) Assume also that $(x' \, p - s' \, q)(y' \, q - t' \, a) - x' \, q \, y' \, p > 0$ for all $(p, q, a)$. Does the equilibrium price of good 1 necessarily increase if $a$ increases?

2.4 Solutions to exercises on differentials

1. a. $dz = (y^2 + 3x^2)dx + 2xy \, dy.$

b. $dz = 2a_1 x_1 dx_1 + \ldots + 2a_n x_n dx_n.$

c. $dz = A(\alpha_1 x_1^{\gamma_1} + \ldots + \alpha_n x_n^{\gamma_n})^{-1+\gamma}(\alpha_1 x_1^{\gamma_1} dx_1 + \ldots + \alpha_n x_n^{\gamma_n} dx_n).$

2. a. Taking differentials of the equations we obtain

$$u' dx + 3wu' du + dv = 2ydy$$
$$3vdu + 3udv - dx = 0$$

b. or, rearranging these equations,

$$3wu' du + dv = -u' dx + 2ydy$$
$$3vdu + 3udv = dx$$

c. or

$$\begin{pmatrix} 3wu' & 1 \\ 3v & 3u \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} -u' dx + 2ydy \\ dx \end{pmatrix}.$$  

d. Inverting the matrix on the left, we obtain $du = -(3w' + 1)/D dx + (6yw/D)dy$ and $dv = (3wu' + 3w'v)/D dx - (6wy/D)dy$, where $D = 9wu' - 3v$.

e. $\partial u/\partial x = -(3u' + 1)/D$ and $\partial v/\partial x = (3wu' + 3w'v)/D$.

3. Taking differentials we obtain

$$U_{i1}''(x,y)dx + U_{i1}''(x,y)dy = p\lambda dx + \lambda dp$$
$$U_{i2}''(x,y)dx + U_{i2}''(x,y)dy = q\lambda dy + \lambda dq$$
$$xdp + pdx + ydq + qdy = dI$$

4. or, setting $dq = dI = 0,$

$$\begin{pmatrix} U_{i1}'' & U_{i1}'' & -p \\ U_{i2}'' & U_{i2}'' & -q \\ p & q & 0 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ d\lambda \end{pmatrix} = \begin{pmatrix} \lambda dp \\ 0 \\ -xdp \end{pmatrix}.$$
5. Using Cramer’s rule we find that
6. \( dx = (\lambda q \cdot q x U'' - px U'') dp / \Delta \),
7. where \( \Delta \) is the determinant of the matrix on the left-hand side of the equation, or
\[
\frac{\partial x}{\partial p} = \frac{\lambda q \cdot x (q x U'' - p U'')}{\Delta}.
\]

8. Taking differentials we obtain
\[
\begin{align*}
    dY &= dC + dI + dG \\
    dC &= f' \cdot (Y, T, r) dY + f'' \cdot (Y, T, r) dT + f' \cdot (Y, T, r) dr \\
    dI &= h' \cdot (Y, r) dY + h' \cdot (Y, r) dr,
\end{align*}
\]

9. or
10. \( dY (1 - f' \cdot (Y, T, r) - h' \cdot (Y, r)) = f' \cdot (Y, T, r) dT + (f' \cdot (Y, T, r) + h' \cdot (Y, r)) dr + dG \),

11. or
\[
\begin{align*}
    dY = f' \cdot (Y, T, r) dT + (f' \cdot (Y, T, r) + h' \cdot (Y, r)) dr + dG \\
        \frac{1}{1 - f' \cdot (Y, T, r) - h' \cdot (Y, r)}
\end{align*}
\]

12. Thus if \( dT = dG \) and \( dr = 0 \) we have
\[
\begin{align*}
    dY &= (1 + f' \cdot (Y, T, r)) dT \\
        \frac{1}{1 - f' \cdot (Y, T, r) - h' \cdot (Y, r)}
\end{align*}
\]

13. (Given that the question asks only about changes in \( T \) and \( G \), we could ignore the change \( dr \) in \( r \) from the beginning.)

14. We have
\[
\begin{align*}
    dq_1 &= f' \cdot (q_1, \alpha) dq_1 + f' \cdot (q_2, \alpha) d\alpha \\
    dq_2 &= g' \cdot (q) dq_1
\end{align*}
\]

15. so that
16. \( \partial q / \partial \alpha = f' \cdot (q_1, \alpha) / [1 - f' \cdot (q_1, \alpha) g' \cdot (q)] \).

17. Since we don’t know if \( f' \cdot g' \) is less than or greater than 1, we don’t know if this is positive or negative.

18. We have
\[
\begin{align*}
    f' \cdot (x) dx &= g' \cdot (y) dy \\
    Ady + ydA + h' \cdot (x) dx &= 0
\end{align*}
\]
19. so that, solving these two equations for $dx$ and $dy$ we deduce that
20. $\frac{\partial x}{\partial A} = -g'(y)\{A f'(x) + g'(y)h'(x)\}$
21. and
22. $\frac{\partial y}{\partial A} = -f''(x)\{A f'(x) + g'(y)h'(x)\}$
23. We have

$$ds_1 = f_1'(s, \alpha)ds_2 + f_2'(s, \alpha)d\alpha$$
$$ds_2 = g_1'(s, \beta)ds_1 + g_2'(s, \beta)d\beta$$

24. so that
25. $\frac{\partial s_2}{\partial \alpha} = f_2'(s, \alpha)/\left[1 - f_1'(s, \alpha)g_1'(s, \beta)\right]$.
26. Since $0 < f_1' < 1$ and $0 < g_1' < 1$ and $f_2' < 0$, an increase in $\alpha$ reduces the
equilibrium value of $s$.
27.
   a. The differentials of the pair of equations are

   $$dq_1 = b_1'(q, c_1)dq_2 + b_2'(q, c_1)dc_1$$
   $$dq_2 = b_1'(q, c_2)dq_1 + b_2'(q, c_2)dc_2.$$  

   b. We may write these equations as

   $$\begin{cases}
   1 & -b_1'(q, c_1) & \{ dq_1 \\ dq_2 \} = \begin{bmatrix} b_2'(q, c_1)dc_1 \end{bmatrix},
   \end{cases}$$

   c. so that

   $$\begin{cases}
   dq_1, dq_2 \end{cases} = (1/\Delta) \begin{bmatrix}
   1 & b_1'(q, c_1) & \{ b_2'(q, c_1)dc_1 \}
   \end{bmatrix},$$

   d. where $\Delta = 1 - b_1'(q, c_1)b_1'(q, c_2)$. Hence

   $dq_1 = (1/\Delta)[b_1'(q, c_1)dc_1 + b_2'(q, c_1)b_2'(q, c_1)dc_1]$ 
   $dq_2 = (1/\Delta)[b_1'(q, c_2)b_2'(q, c_1)dc_1 + b_2'(q, c_2)dc_1].$

   e. To find the effect on $q_1$ and $q_2$ of small and equal increases in $c_1$ and $c_2$, set
   $dc_1 = dc_2 = dc$. Given $q_1 = q_2 = q$ and $c_1 = c_2 = c$ we have $b_2'(q, c_1) =
   b_2'(q, c_2)$, so that

   $dq_1 = (1/\Delta)b_1'(q, c)[1 + b_1'(q, c)]dc$
   $dq_2 = (1/\Delta)b_2'(q, c)[b_2'(q, c) + 1]dc,$
f. so that \( dq_1 = dq_2 < 0 \). That is, both outputs decrease, and do so by the same amount.

28. Taking differentials we obtain

\[
I'(r)dr = S'(Y)dY
\]

\[
adY + L'(r)dr = dM
\]

29. or

\[
\begin{pmatrix}
I'(r) & -S'(Y)
\end{pmatrix} \begin{pmatrix}
dr \\
dY
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

30. Solving for \( dr \) and setting \( dY = 0 \) we get

\[
(\partial r/\partial M) = S'(Y)/[aI'(r) + L'(r)S'(Y)] < 0.
\]

31. Similarly, solving for \( dY \) and setting \( dr = 0 \) we get

\[
(\partial Y/\partial M) = I'(r)/[aI'(r) + L'(r)S'(Y)] > 0.
\]

32. Take differentials of the system of equations (noting that \( p \) and \( q \) are variables, and \( a \) is a parameter):

\[
x'(p, q, a)dp + x'(p, q, a)dq + x'(p, a)da = s'(p)dp
\]

\[
y'(p, q, a)dp + y'(p, q, a)dq + y'(p, a)da = t'(q)dq.
\]

33. Now solve for \( dp \) and \( dq \). Writing the system in matrix form (omitting the arguments of the functions), we have

\[
\begin{pmatrix}
x' - s' \\
x'
\end{pmatrix} \begin{pmatrix}
dp \\
dq
\end{pmatrix} = \begin{pmatrix}
-x'da \\
-x'da
\end{pmatrix},
\]

34. so that

\[
\begin{pmatrix}
dp \\
dq
\end{pmatrix} = \begin{pmatrix}
y' - t' \\
y'
\end{pmatrix} \begin{pmatrix}
-x'da \\
-x'da
\end{pmatrix},
\]

35. where \( \Delta = (x' - s')(y' - t') - x'y' \). Hence

\[
dp = (1/\Delta)[-\begin{pmatrix}
y' - t' \\
y'
\end{pmatrix} \begin{pmatrix}
x'
\end{pmatrix} + \begin{pmatrix}
x'
\end{pmatrix} \begin{pmatrix}
y'
\end{pmatrix}]da.
\]

36. Under the stated assumptions, \( \Delta > 0 \), but the coefficient of \( da \) is not necessarily positive--if \( x' \) and/or \( y' \) are large enough then it could be negative. Thus the price
of good 1 may decrease when \( a \) increases: if the (positive) effect of an increase in \( a \) on the demand for good 1 is small while the (negative) effect on the demand for good 2 is large, then the equilibrium price of good 1 may fall when \( a \) increases.

40. If, however, \( x'_0 y'_0 \) is small then the equilibrium price of good 1 definitely increases.

## 2.5 Homogeneous functions

**Definition**

Multivariate functions that are "homogeneous" of some degree are often used in economic theory. A function is homogeneous of degree \( k \) if, when each of its arguments is multiplied by any number \( t > 0 \), the value of the function is multiplied by \( t^k \). For example, a function is homogeneous of degree 1 if, when all its arguments are multiplied by any number \( t > 0 \), the value of the function is multiplied by the same number \( t \).

Here is a precise definition.

**Definition**

A function \( f \) of \( n \) variables is **homogeneous of degree** \( k \) if 
\[
 f (tx_1, ..., tx_n) = t^k f (x_1, ..., x_n) \text{ for all } (x_1, ..., x_n) \text{ and all } t > 0.
\]

**Example**

For the function \( f (x_1, x_2) = Ax_1 x_2 \) we have 
\[
 f (tx_1, tx_2) = A(tx_1)(tx_2) = t^k A x_1 x_2 = t^k f (x_1, x_2),
\]
so that \( f \) is homogeneous of degree \( a + b \).

**Example**

Let \( f (x_1, x_2) = x_1 + x_2 \). Then 
\[
 f (tx_1, tx_2) = tx_1 + tx_2.
\]

It doesn't seem to be possible to write this expression in the form \( t^k (x_1 + x_2) \) for any value of \( k \). But how do we prove that there is no such value of \( k \)? Suppose that there were such a value. That is, suppose that for some \( k \) we have 
\[
 tx_1 + tx_2 = t^k (x_1 + x_2) \text{ for all } (x_1, x_2) \text{ and all } t > 0.
\]

Then in particular, taking \( t = 2 \), we have 
\[
 2x_1 + 4x_2 = 2^k (x_1 + x_2) \text{ for all } (x_1, x_2).
\]

Taking \( (x_1, x_2) = (1, 0) \) and \( (x_1, x_2) = (0, 1) \) we thus have 
\[
 2 = 2^k \text{ and } 4 = 2^k,
\]
which is not possible. Thus \( f \) is not homogeneous of any degree.

In economic theory we often assume that a firm's production function is homogeneous of degree 1 (if all inputs are multiplied by \( t \) then output is multiplied by \( t \)). A production function with this property is said to have "constant returns to scale".

Suppose that a consumer's demand for goods, as a function of prices and her income, arises from her choosing, among all the bundles she can afford, the one that is best according to her preferences. Then we can show that this demand function is homogeneous of degree zero: if all prices and the consumer's income are multiplied by any number \( t > 0 \) then her demands for goods stay the same.
Partial derivatives of homogeneous functions

The following result is sometimes useful.

**Proposition**

Let $f$ be a differentiable function of $n$ variables that is homogeneous of degree $k$. Then each of its partial derivatives $f_i'$ (for $i = 1, \ldots, n$) is homogeneous of degree $k - 1$.

**Proof**

The homogeneity of $f$ means that $f(tx_1, \ldots, tx_n) = t^k f(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n)$ and all $t > 0$. Now differentiate both sides of this equation with respect to $x_i$ to get $t f_i'(tx_1, \ldots, tx_n) = t^k f_i'(x_1, \ldots, x_n)$, and then divide both sides by $t$ to get $f_i'(tx_1, \ldots, tx_n) = t^{k-1} f_i'(x_1, \ldots, x_n)$, so that $f_i'$ is homogeneous of degree $k - 1$.

**Application: level curves of homogeneous functions**

This result can be used to demonstrate a nice result about the slopes of the level curves of a homogeneous function. As we have seen, the slope of the level curve of the function $F$ through the point $(x_0, y_0)$ at this point is

$$-rac{F_1'(x_0, y_0)}{F_2'(x_0, y_0)}.$$

Now suppose that $F$ is homogeneous of degree $k$, and consider the level curve through $(cx_0, cy_0)$ for some number $c > 0$. At $(cx_0, cy_0)$, the slope of this curve is

$$-rac{F_1'(cx_0, cy_0)}{F_2'(cx_0, cy_0)}.$$

By the previous result, $F_1'$ and $F_2'$ are homogeneous of degree $k-1$, so this slope is equal to

$$\frac{c^{k-1}F_1'(x_0, y_0)}{c^{k-1}F_2'(x_0, y_0)} = \frac{F_1'(x_0, y_0)}{F_2'(x_0, y_0)}.$$

That is, the slope of the level curve through $(cx_0, cy_0)$ at the point $(cx_0, cy_0)$ is exactly the same as the slope of the level curve through $(x_0, y_0)$ at the point $(x_0, y_0)$, as illustrated in the following figure.
In this figure, the red lines are two level curves, and the two green lines, the tangents to the curves at \((x_0, y_0)\) and at \((cx_0, xy_0)\), are parallel.

We may summarize this result as follows.

Let \(F\) be a differentiable function of two variables that is homogeneous of some degree. Then along any given ray from the origin, the slopes of the level curves of \(F\) are the same.

**Euler's theorem**

A function homogeneous of some degree has a property sometimes used in economic theory that was first discovered by Leonhard Euler (1707-1783).

**Proposition (Euler's theorem)**

The differentiable function \(f\) of \(n\) variables is homogeneous of degree \(k\) if and only if

\[
\sum_{i=1}^{n} x_i f_i'(x_1, ..., x_n) = kf(x_1, ..., x_n)
\]

for all \((x_1, ..., x_n)\).

### 3.1 Concave and convex functions of a single variable

**General definitions**

The twin notions of concavity and convexity are used widely in economic theory, and are also central to optimization theory. A function of a single variable is *concave* if every line segment joining two points on its graph does not lie above the graph at any point. Symmetrically, a function of a single variable is *convex* if every line segment joining two points on its graph does not lie below the graph at any point. These concepts are illustrated in the following figure.
Here is a precise definition.

**Definition**

Let $f$ be a function of a single variable defined on an interval. Then $f$ is

- **concave** if every line segment joining two points on its graph is never above the graph
- **convex** if every line segment joining two points on its graph is never below the graph.

To make this definition useful we need to translate it into an algebraic condition that we can check. Let $f$ be a function defined on the interval $[x_1, x_2]$. This function is concave according to the definition if, for every pair of numbers $a$ and $b$ with $x_1 \leq a \leq x_2$ and $x_1 \leq b \leq x_2$, the line segment from $(a, f(a))$ to $(b, f(b))$ lies on or below the function, as illustrated in the following figure.

Denote the height of the line segment from $(a, f(a))$ to $(b, f(b))$ at the point $x$ by $h_{ab}(x)$. Then for the function $f$ to be concave, we need

$$f(x) \geq h_{ab}(x) \text{ for all } x \text{ with } a \leq x \leq b \quad (*)$$

for every pair of numbers $a$ and $b$ with $x_1 \leq a \leq x_2$ and $x_1 \leq b \leq x_2$. 
Now, every point \( x \) with \( a \leq x \leq b \) may be written as \( x = (1 - \lambda)a + \lambda b \), where \( \lambda \) is a real number from 0 to 1. (When \( \lambda = 0 \), we have \( x = a \); when \( \lambda = 1 \) we have \( x = b \).) The fact that \( h_{a,b} \) is linear means that

\[
h_{a,b}(1 - \lambda)a + \lambda b) = (1 - \lambda)h_{a,b}(a) + \lambda h_{a,b}(b)
\]

for any value of \( \lambda \) with \( 0 \leq \lambda \leq 1 \). Further, we have \( h_{a,b}(a) = f(a) \) and \( h_{a,b}(b) = f(b) \) (the line segment coincides with the function at its endpoints), so

\[
h_{a,b}(1 - \lambda)a + \lambda b) = (1 - \lambda)f(a) + \lambda f(b).
\]

Thus the condition (*) is equivalent to

\[
f((1-\lambda)a + \lambda b) \geq (1 - \lambda)f(a) + \lambda f(b)
\]

for all \( \lambda \) with \( 0 \leq \lambda \leq 1 \).

We can make a symmetric argument for a convex function. Thus the definition of concave and convex functions may be rewritten as follows.

**Definition**

Let \( f \) be a function of a single variable defined on the interval \( I \). Then \( f \) is

- **concave** if for all \( a \in I \), all \( b \in I \), and all \( \lambda \in (0, 1) \) we have
  \[
f((1-\lambda)a + \lambda b) \geq (1 - \lambda)f(a) + \lambda f(b)
\]
- **convex** if for all \( a \in I \), all \( b \in I \), and all \( \lambda \in (0, 1) \) we have
  \[
f((1-\lambda)a + \lambda b) \leq (1 - \lambda)f(a) + \lambda f(b).
\]

In an exercise you are asked to show that \( f \) is convex if and only if \( -f \) is concave.

Note that a function may be both concave and convex. Let \( f \) be such a function. Then for all values of \( a \) and \( b \) we have

\[
f((1-\lambda)a + \lambda b) \geq (1 - \lambda)f(a) + \lambda f(b)
\]

and

\[
f((1-\lambda)a + \lambda b) \leq (1 - \lambda)f(a) + \lambda f(b)
\]

for all \( \lambda \in (0, 1) \). Equivalently, for all values of \( a \) and \( b \) we have

\[
f((1-\lambda)a + \lambda b) = (1 - \lambda)f(a) + \lambda f(b)
\]

for all \( \lambda \in (0, 1) \). That is, a function is both concave and convex if and only if it is linear (or, more precisely, affine), taking the form \( f(x) = \alpha + \beta x \) for all \( x \), for some constants \( \alpha \) and \( \beta \).

Economists often assume that a firm's production function is increasing and concave. An example of such a function for a firm that uses a single input is shown in the next figure. The fact that such a production function is increasing means that more input generates more output. The fact that it is concave means that the increase in output generated by a one-unit increase in the input is smaller when output is large than when it is small. That
is, there are "diminishing returns" to the input, or, given that the firm uses a single input, "diminishing returns to scale". For some (but not all) production processes, this property seems reasonable.

The notions of concavity and convexity are important in optimization theory because, as we shall see, the first-order conditions are sufficient (as well as necessary) for a maximizer of a concave function and for a minimizer of a convex function. (Precisely, every point at which the derivative of a concave differentiable function is zero is a maximizer of the function, and every point at which the derivative of a convex differentiable function is zero is a minimizer of the function.)

The next example shows that a nondecreasing concave transformation of a concave function is concave.

**Example**

Let $U$ be a concave function and $g$ a nondecreasing and concave function. Define the function $f$ by $f(x) = g(U(x))$ for all $x$. Show that $f$ is concave.

We need to show that $f((1-\lambda)a + \lambda b) \geq (1-\lambda)f(a) + \lambda f(b)$ for all values of $a$ and $b$ with $a \leq b$.

By the definition of $f$ we have

$$f((1-\lambda)a + \lambda b) = g(U((1-\lambda)a + \lambda b)).$$

Now, because $U$ is concave we have

$$U((1-\lambda)a + \lambda b) \geq (1-\lambda)U(a) + \lambda U(b).$$

Further, because $g$ is nondecreasing, $r \geq s$ implies $g(r) \geq g(s)$. Hence

$$g(U((1-\lambda)a + \lambda b)) \geq g((1-\lambda)U(a) + \lambda U(b)).$$

But now by the concavity of $g$ we have

$$g((1-\lambda)U(a) + \lambda U(b)) \geq (1-\lambda)g(U(a)) + \lambda g(U(b)) = (1-\lambda)f(a) + \lambda f(b).$$

So $f$ is concave.
Twice-differentiable functions
We often assume that the functions in economic models (e.g. a firm's production function, a consumer's utility function) are differentiable. We may determine the concavity or convexity of a twice differentiable function (i.e. a function that is differentiable and that has a differentiable derivative) by examining its second derivative: a function whose second derivative is nonpositive everywhere is concave, and a function whose second derivative is nonnegative everywhere is convex.

Here is a precise result.

**Proposition**
A twice-differentiable function \( f \) of a single variable defined on the interval \( I \) is

- concave if and only if \( f''(x) \leq 0 \) for all \( x \) in the interior of \( I \)
- convex if and only if \( f''(x) \geq 0 \) for all \( x \) in the interior of \( I \).

The importance of concave and convex functions in optimization theory comes from the fact that if the differentiable function \( f \) is concave then every point \( x \) at which \( f'(x) = 0 \) is a global maximizer, and if it is convex then every such point is a global minimizer.

**Example**
Is \( x^3 - 2x + 2 \) concave or convex on any interval? Its second derivative is \( 2 \geq 0 \), so it is convex for all values of \( x \).

**Example**
Is \( x^2 - x^3 \) concave or convex on any interval? Its second derivative is \( 6x - 2 \), so it is convex on the interval \([1/3, \infty)\) and concave the interval \((-\infty, 1/3]\).

The next example shows how the result in an earlier example may be established for twice-differentiable functions. (The earlier result is true for all functions, so the example proves a result we already know to be true; it is included only to show how a version of the earlier result for twice-differentiable functions may be established by using the characterization of concavity in the previous Proposition.)

**Example**
Let \( U \) be a concave function and \( g \) a nondecreasing and concave function. Assume that \( U \) and \( g \) are twice-differentiable. Define the function \( f \) by \( f(x) = g(U(x)) \) for all \( x \). Show that \( f \) is concave.

We have \( f'(x) = g'(U(x))U'(x) \), so that

\[
f''(x) = g''(U(x))U'(x)^2 + g'(U(x))U''(x).
\]

Since \( g''(x) \leq 0 \) (\( g \) is concave), \( g'(x) \geq 0 \) (\( g \) is nondecreasing), and \( U''(x) \leq 0 \) (\( U \) is concave), we have \( f''(x) \leq 0 \). That is, \( f \) is concave.
A point at which a twice-differentiable function changes from being convex to concave, or vice versa, is an inflection point.

**Definition**

A point at which a twice-differentiable function changes from being convex to concave, or vice versa, is called an inflection point.

**Proposition**

- If \( c \) is an inflection point of \( f \) then \( f''(c) = 0 \).
- If \( f''(c) = 0 \) and \( f'' \) changes sign at \( c \) then \( c \) is an inflection point.

Note, however, that \( f'' \) does not have to change sign at \( c \) for \( c \) to be an inflection point of \( f \). For example, every point is an inflection point of a linear function.

**Strict convexity and concavity**

The inequalities in the definition of concave and convex functions are weak; such functions may have linear parts, as in the following figure.
A concave function that has no linear parts is said to be strictly concave.

**Definition**

The function $f$ of a single variable defined on the interval $I$ is

- **strictly concave** if for all $a \in I$, all $b \in I$ with $a \neq b$, and all $\lambda \in (0,1)$ we have
  \[
  f((1-\lambda)a + \lambda b) > (1-\lambda)f(a) + \lambda f(b).
  \]

- **strictly convex** if for all $a \in I$, all $b \in I$ with $a \neq b$, and all $\lambda \in (0,1)$ we have
  \[
  f((1-\lambda)a + \lambda b) < (1-\lambda)f(a) + \lambda f(b).
  \]

An earlier result states that if $f$ is twice differentiable then $f$ is concave on $[a, b]$ if and only if $f''(x) \leq 0$ for all $x \in (a, b)$.

Does this result have an analogue for strictly concave functions? Not exactly. If $f''(x) < 0$ for all $x \in (a, b)$ then $f$ is strictly concave on $[a, b]$, but the converse is not true: if $f$ is strictly concave then its second derivative is not necessarily negative at all points.

(Consider the function $f(x) = -x^4$. It is concave, but its second derivative at 0 is zero, not negative.) That is, $f$ is strictly concave on $[a, b]$ if $f''(x) < 0$ for all $x \in (a, b)$, but if $f$ is strictly concave on $[a, b]$ then $f''(x)$ is not necessarily negative for all $x \in (a, b)$.

(Analogous observations apply to the case of convex and strictly convex functions, with the conditions $f''(x) \geq 0$ and $f''(x) > 0$ replacing the conditions $f''(x) \leq 0$ and $f''(x) < 0$.)

### 3.1 Exercises on concave and convex functions of a single variable

1. Show that the function $f$ is convex if and only if the function $-f$ is concave. [Do not assume that the function $f$ is differentiable. The value of the function $-f$ at any point $x$ is $-f(x).$]
2. The functions $f$ and $g$ are both concave functions of a single variable. *Neither function is necessarily differentiable.*
   a. Is the function $h$ defined by $h(x) = f(x) + g(x)$ necessarily concave, necessarily convex, or not necessarily either?
   b. Is the function $h$ defined by $h(x) = -f(x)$ necessarily concave, necessarily convex, or not necessarily either?
   c. Is the function $h(x) = f(x)g(x)$ necessarily concave, necessarily convex, or not necessarily either?
In questions like this, it is probably helpful to first draw some diagrams to get an idea of whether each claim is true or false. If you think a claim is false, try to find an example that is inconsistent with it. For instance, if you think that the first claim in the first part of this question is false---that is, you think that \( h \) is not necessarily concave---then try to find two functions whose sum is not concave. If you think a claim is true, try to prove it. To start a proof, write down precise statements of what you know---in this case, the precise conditions that \( f \) and \( g \) satisfy, given they are concave. Then write down on a separate piece of paper, or at the bottom of the page, a precise statement of the conclusion you want to reach---in this case, the precise condition under which \( h \) is concave. Now you need to figure out a way to get from the first set of conditions to the second set of conditions in logical steps.

3. The function \( f(x) \) is concave, but not necessarily differentiable. Find the values of the constants \( a \) and \( b \) for which the function \( af(x) + b \) is concave. (Give a complete argument; no credit for an argument that applies only if \( f \) is differentiable.)

4. The function \( g \) of a single variable is defined by \( g(x) = f(ax + b) \), where \( f \) is a concave function of a single variable that is not necessarily differentiable, and \( a \) and \( b \) are constants with \( a \neq 0 \). (These constants may be positive or negative.) Either show that the function \( g \) is concave, or show that it is not necessarily concave. [Your argument must apply to the case in which \( f \) is not necessarily differentiable.]

5. Determine the concavity/convexity of \( f(x) = -(1/3)x^2 + 8x - 3 \).

6. Let \( f(x) = Ax^\alpha \), where \( A > 0 \) and \( \alpha \) are parameters. For what values of \( \alpha \) is \( f \) (which is twice differentiable) nondecreasing and concave on the interval \([0, \infty)\)?

7. Find numbers \( a \) and \( b \) such that the graph of the function \( f(x) = ax^3 + bx^2 \) passes through \((-1, 1)\) and has an inflection point at \( x = 1/2 \).

8. A competitive firm receives the price \( p > 0 \) for each unit of its output, and pays the price \( w > 0 \) for each unit of its single input. Its output from using \( x \) units of the variable input is \( f(x) = x^\alpha \). Is this production function concave? Is the firm’s profit concave in \( x \)?

9. A firm has the continuous production function \( f \), sells output at the price \( p \), and pays \( w_i \) per unit for input \( i \), \( i = 1, ..., L \). Consider its profit maximization problem

\[
\max z \cdot p f(z) - wz \quad \text{subject to } z \geq 0.
\]

Fix the value of the vector \( w \). Assume that for each value of \( p \) the problem has a solution \( z^*(p) \) with \( z^*(p) > 0 \). Let

\[
\pi(p) = pf(z^*(p)) - wz^*(p),
\]

the maximal profit of the firm when the price is \( p \). Show that \( \pi \) is a convex function of \( p \). [Hint: Let \( p \) and \( p' \) be arbitrary values of the parameter \( p \), and let \( p'' = (1-\alpha)p + \alpha p' \) (for convenience). We have \( \pi(p'') = p'' f(z^*(p'')) - wz^*(p'') \). How are \( pf(z^*(p'')) - wz^*(p'') \) and \( p' f(z^*(p'')) - wz^*(p'') \) related to \( \pi(p) \) and \( \pi(p') \)?]
3.1 Solutions to exercises on concave and convex functions

1. First suppose that the function $f$ is convex. Then for all values of $a$ and $b$ with $a \leq b$ we have

$$f((1−\lambda)a + \lambda b) \leq (1 − \lambda)f(a) + \lambda f(b).$$

Multiply both sides of this equation by $−1$ (which changes the inequality):

$$−f((1−\lambda)a + \lambda b) \geq −[(1 − \lambda)f(a) + \lambda f(b)],$$
or

$$−f((1−\lambda)a + \lambda b) \geq (1 − \lambda)(−f(a)) + \lambda(−f(b)).$$

Thus $−f$ is concave.

Now suppose that the function $−f$ is concave. Then

$$−f((1−\lambda)a + \lambda b) \geq (1 − \lambda)(−f(a)) + \lambda(−f(b)).$$

Multiplying both sides of this equation by $−1$, gives

$$f((1−\lambda)a + \lambda b) \leq (1 − \lambda)f(a) + \lambda f(b),$$
so that $f$ is convex.

2. 

a. We have

$$h(\alpha x + (1−\alpha)y) = f(\alpha x + (1−\alpha)y) + g(\alpha x + (1−a)y)$$

$$\geq \alpha f(x) + (1 − \alpha) f(y) + \alpha g(x) + (1 − \alpha)g(y)$$

(using the concavity of $f$ and of $g$)

$$= \alpha(f(x) + g(x)) + (1−\alpha)(f(y) + g(y))$$

$$= \alpha h(x) + (1−a)h(y).$$

b. Thus $h$ is necessarily concave.

c. Since $f$ is concave, we have

$$f(\alpha x + (1−\alpha)y) \geq \alpha f(x) + (1−\alpha) f(y) \text{ for all } x, y, \text{ and } \alpha.$$ 

Hence

$$−f(\alpha x + (1−\alpha)y) \leq \alpha(−f(x)) + (1−\alpha)(−f(y)) \text{ for all } x, y, \text{ and } \alpha,$$

so that $−f$ is convex.

d. The function $h$ is neither necessarily concave nor necessarily convex. If $f(x) = x$ and $g(x) = x$ then both $f$ and $g$ are concave, but $h$ is convex and not concave. Thus $h$ is not necessarily concave. If $f(x) = x$ and $g(x) = −x$ then both $f$ and $g$ are concave, and $h$ is strictly concave, and hence not convex. Thus $h$ is not necessarily convex.

3. Let $g(x) = af(x) + b$. Because $f$ is concave we have

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We have

\[ g((1-\alpha)x + \alpha y) = a f((1-\alpha)x + \alpha y) + b \]

and

\[ (1-\alpha)g(x) + \alpha g(y) = a[(1-\alpha)f(x) + \alpha f(y)] + b. \]

Thus

\[ g((1-\alpha)x + \alpha y) \geq (1-\alpha)g(x) + \alpha g(y) \] for all \( x, y, \) and \( \alpha \in [0,1] \)

if and only if

\[ a f((1-\alpha)x + \alpha y) \geq a((1-\alpha)f(x) + \alpha f(y)), \]

or if and only if \( a \geq 0 \) (using the concavity of \( f \)).

4. We have

\[ g(\alpha x_i + (1-\alpha)x_j) = f(\alpha x_i + (1-\alpha)x_j) + b \]

\[ = f(\alpha(ax_i + b) + (1-\alpha)(ax_j + b)) \]

\[ \geq \alpha f(ax_i + b) + (1-\alpha)f(ax_j + b) \]

(by the concavity of \( f \))

\[ = \alpha g(x_i) + (1-\alpha)g(x_j). \]

5. Thus \( g \) is concave.

6. The function is twice-differentiable, because it is a polynomial. We have \( f'(x) = -2x/3 + 8 \) and \( f''(x) = -2/3 < 0 \) for all \( x \), so \( f \) is strictly concave.

7. We have \( f'(x) = \alpha Ax^{\alpha-1} \) and \( f''(x) = \alpha(\alpha-1)Ax^{\alpha-2} \). For any value of \( \beta \) we have \( x^\beta \geq 0 \) for all \( x \geq 0 \), so for \( f \) to be nondecreasing and concave we need \( \alpha \geq 0 \) and \( \alpha(\alpha - 1) \leq 0 \), or equivalently \( 0 \leq \alpha \leq 1 \).

8. For the graph of the function to pass through \((-1,1)\) we need \( f(-1) = 1 \), which implies that \(-a + b = 1\). Now, we have \( f'(x) = 3ax^2 + 2bx \) and \( f''(x) = 6ax + 2b \), so for \( f \) to have an inflection point at \( 1/2 \) we need \( f''(1/2) = 0 \), which yields \( 3a + 2b = 0 \). Solving these two equations in \( a \) and \( b \) yields \( a = -2/5 \), \( b = 3/5 \).

9. The function \( f \) is twice-differentiable for \( x > 0 \). We have \( f'(x) = (1/4)x^{-3/4} \) and \( f''(x) = -(3/16)x^{-7/4} < 0 \) for all \( x \), so \( f \) is concave for \( x > 0 \). It is continuous, so it is concave for all \( x \geq 0 \). The firm’s profit, \( pf(x) - wx \), is thus the sum of two concave functions, and is hence concave.

10. We need to show that \( \pi(p') \leq (1-\alpha)\pi(p) + \alpha\pi(p') \) for all \( p, p' \) and all \( \alpha \in [0,1] \).

(Remember that \( p'' = (1-\alpha)p + \alpha p' \)). Now,

\[ \pi(p'') = p'' f(z^*(p'')) - w^*z^*(p'') \]

\[ = (1-\alpha)[pf(z^*(p'')) - w^*z^*(p'')] + \alpha[p' f(z^*(p'')) - w^*z^*(p'')]. \]

11. Further, \( \pi(p) \geq pf(z) - wz \) for every value of \( z \), by the definition of \( \pi(p) \), which is the maximal profit of the firm when the price is \( p \). Thus, choosing \( z = z^*(p'') \), we have \( \pi(p) \geq pf(z^*(p'')) - w^*z^*(p'') \). Similarly, \( \pi(p') \geq p' f(z^*(p'')) - w^*z^*(p'') \).

Hence \( \pi(p'') \leq (1-\alpha)\pi(p) + \alpha\pi(p') \).
A quadratic form in many variables is the sum of several terms, each of which is a constant times the product of exactly two variables.

**Definition**

A **quadratic form** in $n$ variables is a function

$$Q(x_1, ..., x_n) = b_{11}x_1^2 + b_{12}x_1x_2 + ... + b_{ij}x_ix_j + ... + b_{nn}x_n^2$$

where $b_{ij}$ for $i = 1, ..., n$ and $j = 1, ..., n$ are constants.

**Example**

The function

$$Q(x_1, x_2) = 2x_1^2 + 4x_1x_2 - 6x_2x_1 - 3x_2^2$$

is a quadratic form in two variables.

We can write the quadratic form in this example as

$$Q(x_1, x_2) = (x_1, x_2) \cdot \begin{pmatrix} 2 & 4 \\ 4 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$  

Because $4x_1x_2 - 6x_2x_1 = -2x_1x_2$, we can alternatively write it as

$$Q(x_1, x_2) = (x_1, x_2) \cdot \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$  

In this way of writing the quadratic form, the matrix is *symmetric*.

We can in fact write any quadratic form as

$$Q(x) = x'Ax$$

where $x$ is the column vector of $x$'s and $A$ is a symmetric $n \times n$ matrix for which the $(i, j)$th element is $a_{ij} = (1/2)(b_{ij} + b_{ji})$. The reason is that $x_ix_j = x_jx_i$ for any $i$ and $j$, so that

$$b_{ij}x_i + b_{ji}x_j = (b_{ij} + b_{ji})x_ix_j$$

$$= (1/2)(b_{ij} + b_{ji})x_jx_i + (1/2)(b_{ji} + b_{ij})x_ix_j.$$

**Example**

Let $Q(x_1, x_2, x_3) = 3x_1^2 + 3x_1x_3 - x_3x_1 + 3x_2x_1 + x_2x_3 + 2x_3x_1 + 4x_2x_3 - x_2^2 + 2x_3^2$. That is, $a_{11} = 3$, $a_{13} = 3$, $a_{33} = -1$, etc. We have $Q(x) = x'Ax$ where

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 3 & 2 \end{pmatrix}.$$
Subsequently, when representing a quadratic form as \( x'Ax \), we will \textbf{always} take the matrix \( A \) to be symmetric.

### 3.2.2 Quadratic forms: conditions for definiteness

**Definitions**

Relevant questions when we use quadratic forms in studying the concavity and convexity of functions of many variables are:

- Under what condition on the matrix \( A \) are the values of the quadratic form \( Q(x) = x'Ax \) positive for all values of \( x \neq 0 \)?
- Under what condition are these values negative for all values of \( x \neq 0 \)?

The following terminology is useful.

**Definition**

Let \( Q(x) \) be a quadratic form, and let \( A \) be the symmetric matrix that represents it (i.e. \( Q(x) = x'Ax \)). Then \( Q(x) \) (and the associated matrix \( A \)) is

- **positive definite** if \( x'Ax > 0 \) for all \( x \neq 0 \)
- **negative definite** if \( x'Ax < 0 \) for all \( x \neq 0 \)
- **positive semidefinite** if \( x'Ax \geq 0 \) for all \( x \)
- **negative semidefinite** if \( x'Ax \leq 0 \) for all \( x \)
- **indefinite** if it is neither positive nor negative semidefinite (i.e. if \( x'Ax > 0 \) for some \( x \) and \( x'Ax < 0 \) for some \( x \)).

**Example**

\[ x_1^2 + x_2^2 > 0 \text{ if } (x_1, x_2) \neq 0, \] so this quadratic form is positive definite. More generally, \( ax_1^2 + cx_2^2 \) is positive definite whenever \( a > 0 \) and \( c > 0 \).

**Example**

\( (x_1 + x_2)^2 \geq 0 \text{ for all } (x_1, x_2), \) so that this quadratic form is positive semidefinite. It is not positive definite because \( (x_1 + x_2)^2 = 0 \text{ for } (x_1, x_2) = (1, -1) \) (for example).

**Example**

\[ x_1^2 - x_2^2 > 0 \text{ for } (x_1, x_2) = (1, 0) \text{ (for example), and } x_1^2 - x_2^2 < 0 \text{ for } (x_1, x_2) = (0, 1) \text{ (for example). Thus this quadratic form is indefinite.} \]

**Two variables**

We can easily derive conditions for the definiteness of any quadratic form in two variables. To make the argument more readable, I change the notation slightly, using \( x \) and \( y \) for the variables, rather than \( x_1 \) and \( x_2 \). Consider the quadratic form \( Q(x, y) = ax^2 + 2bxy + cy^2 \).

If \( a = 0 \) then \( Q(1,0) = 0 \), so \( Q \) is neither positive nor negative definite. So assume that \( a \neq 0 \).
Given $a \neq 0$, we have

\[ Q(x, y) = a[(x + (b/a)y)^2 + (c/a - (b/a)^2)y^2]. \]

Both squares are always nonnegative, and at least one of them is positive unless $(x, y) = (0, 0)$. Thus if $a > 0$ and $c/a - (b/a)^2 > 0$ then $Q(x, y)$ is positive definite. Given $a > 0$, the second condition is $ac > b^2$. Thus we conclude that if $a > 0$ and $ac > b^2$ then $Q(x, y)$ is positive definite.

Now, we have $Q(1, 0) = a$ and $Q(-b/a, 1) = (ac - b^2)/a$. Thus, if $Q(x, y)$ is positive definite then $a > 0$ and $ac > b^2$.

We conclude that $Q(x, y)$ is positive definite if and only if $a > 0$ and $ac > b^2$.

A similar argument shows that $Q(x, y)$ is negative definite if and only if $a < 0$ and $ac > b^2$.

Note that if $a > 0$ and $ac > b^2$ then because $b^2 \geq 0$ for all $b$, we can conclude that $c > 0$. Similarly, if $a < 0$ and $ac > b^2$ then $c < 0$. Thus, to determine whether a quadratic form is positive or negative definite we need to look only at the signs of $a$ and of $ac - b^2$; but if the conditions for positive definiteness are satisfied then it must in fact also be true that $c > 0$, and if the conditions for negative definiteness are satisfied then we must also have $c < 0$.

Notice that $ac - b^2$ is the determinant of the matrix that represents the quadratic form, namely

\[ A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \]

Thus we can rewrite the results as follows: the two variable quadratic form $Q(x, y) = ax^2 + 2bxy + cy^2$ is

- positive definite if and only if $a > 0$ and $|A| > 0$ (in which case $c > 0$)
- negative definite if and only if $a < 0$ and $|A| > 0$ (in which case $c < 0$)

Many variables

To obtain conditions for an $n$-variable quadratic form to be positive or negative definite, we need to examine the determinants of some of its submatrices.

**Definition**

The $k$th order leading principal minor of the $n \times n$ symmetric matrix $A = (a_{ij})$ is the determinant of the matrix obtained by deleting the last $n - k$ rows and columns of $A$ (where $k = 1, \ldots, n$):

\[ D_k = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{vmatrix} \]
Let \( A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 3 & 2 \end{pmatrix} \).

The first-order leading principal minor \( D_1 \) is the determinant of the matrix obtained from \( A \) by deleting the last two rows and columns; that is, \( D_1 = 3 \). The second-order leading principal minor \( D_2 \) is the determinant of the matrix obtained from \( A \) by deleting the last row and column; that is,

\[
D_2 = \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix},
\]

so that \( D_2 = -4 \). Finally, the third-order leading principal minor \( D_3 \) is the determinant of \( A \), namely \(-19\).

The following result characterizes positive and negative definite quadratic forms (and their associated matrices).

**Proposition**

Let \( A \) be an \( n \times n \) symmetric matrix and let \( D_k \) for \( k = 1, \ldots, n \) be its leading principal minors. Then

- \( A \) is positive definite if and only if \( D_k > 0 \) for \( k = 1, \ldots, n \).
- \( A \) is negative definite if and only if \( (-1)^k D_k > 0 \) for \( k = 1, \ldots, n \). (That is, if and only if the leading principal minors alternate in sign, starting with negative for \( D_1 \).

In the special case that \( n = 2 \) these conditions reduce to the previous ones because for

\[
A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}
\]

we have \( D_1 = a \) and \( D_2 = ac - b^2 \).

**Example**

Let \( A = \begin{pmatrix} -3 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & -5 \end{pmatrix} \).

The leading principal minors of \( A \) are \( D_1 = -3 < 0 \), \( D_2 = (-3)(-3)-(2)(2) = 5 > 0 \), and \( |A| = -25 < 0 \). Thus \( A \) is negative definite.
Example

We saw above that the leading principal minors of the matrix

\[
A = \begin{pmatrix}
3 & 1 & 2 \\
1 & -1 & 3 \\
2 & 3 & 2
\end{pmatrix}
\]

are \(D_1 = 3\), \(D_2 = -4\), and \(D_3 = -19\). Thus \(A\) is neither positive definite nor negative definite. (Note that we can tell this by looking only at the first two leading principal minors---there is no need to calculate \(D_3\).)

### 3.2.3 Quadratic forms: conditions for semidefiniteness

#### Two variables

First consider the case of a two-variable quadratic form \(Q(x, y) = ax^2 + 2bxy + cy^2\).

If \(a = 0\) then \(Q(x, 1) = 2b + c\). This expression is nonnegative for all values of \(x\) if and only if \(b = 0\) and \(c \geq 0\), in which case \(ac - b^2 = 0\).

Now assume \(a \neq 0\). As before, we have

\[
Q(x, y) = a[(x + (b/a)y)^2 + (c/a - (b/a)^2)y^2].
\]

Both squares are nonnegative, so if \(a > 0\) and \(ac - b^2 \geq 0\) then this expression is nonnegative for all \((x, y)\). If these two conditions are satisfied then \(c \geq 0\).

We conclude that if \(a \geq 0\), \(c \geq 0\), and \(ac - b^2 \geq 0\), then the quadratic form is positive semidefinite.

Conversely, if the quadratic form is positive semidefinite then \(Q(1, 0) = a \geq 0\), \(Q(0, 1) = c \geq 0\), and \(Q(-b, a) = a(ac - b^2) \geq 0\). If \(a = 0\) then by the previous argument we need \(b = 0\) and \(c \geq 0\) in order for the quadratic form to be positive semidefinite, so that \(ac - b^2 = 0\); if \(a > 0\) then we need \(ac - b^2 \geq 0\) in order for \(a(ac - b^2) \geq 0\).

We conclude that the quadratic form is positive semidefinite if and only if \(a \geq 0\), \(c \geq 0\), and \(ac - b^2 \geq 0\).

A similar argument implies that the quadratic form is negative semidefinite if and only if \(a \leq 0\), \(c \leq 0\), and \(ac - b^2 \geq 0\).

Note that in this case, unlike the case of positive and negative definiteness, we need to check all three conditions, not just two of them. If \(a \geq 0\) and \(ac - b^2 \geq 0\), it is not necessarily the case that \(c \geq 0\) (try \(a = b = 0\) and \(c < 0\)), so that the quadratic form is not
necessarily positive semidefinite. (Similarly, the conditions $a \leq 0$ and $ac - b^2 \geq 0$ are not sufficient for the quadratic form to be negative semidefinite: we need, in addition, $c \leq 0$.)

Thus we can rewrite the results as follows: the two variable quadratic form $Q(x, y) = ax^2 + 2bxy + cy^2$ is

- positive semidefinite if and only if $a \geq 0$, $c \geq 0$, and $|A| \geq 0$
- negative semidefinite if and only if $a \leq 0$, $c \leq 0$, and $|A| \geq 0$

where

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

It follows that the quadratic form is indefinite if and only if $|A| < 0$. (Note that if $|A| \geq 0$ then $ac \geq 0$, so we cannot have $a < 0$ and $c > 0$, or $a > 0$ and $c < 0$.)

**Many variables**

As in the case of two variables, to determine whether a quadratic form is positive or negative semidefinite we need to check more conditions than we do in order to check whether it is positive or negative definite. In particular, it is **not** true that a quadratic form is positive or negative semidefinite if the inequalities in the conditions for positive or negative definiteness are satisfied weakly. In order to determine whether a quadratic form is positive or negative semidefinite we need to look at more than simply the **leading** principal minors. The matrices we need to examine are described in the following definition.

**Definition**

The *kth order principal minors* of an $n \times n$ symmetric matrix $A$ are the determinants of the $k \times k$ matrices obtained by deleting $n - k$ rows and the corresponding $n - k$ columns of $A$ (where $k = 1, \ldots, n$).

Note that the $k$th order leading principal minor of a matrix is one of its $k$th order principal minors.

**Example**

Let

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

The first-order principal minors of $A$ are $a$ and $c$, and the second-order principal minor is the determinant of $A$, namely $ac - b^2$.

**Example**

Let

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \end{pmatrix}$$
This matrix has 3 first-order principal minors, obtained by deleting

- the last two rows and last two columns
- the first and third rows and the first and third columns
- the first two rows and first two columns

which gives us simply the elements on the main diagonal of the matrix: 3, \(-1\), and 2. The matrix also has 3 second-order principal minors, obtained by deleting

- the last row and last column
- the second row and second column
- the first row and first column

which gives us \(-4\), 2, and \(-11\). Finally, the matrix has one third-order principal minor, namely its determinant, \(-19\).

The following result gives criteria for semidefiniteness.

**Proposition**

Let \( A \) be an \( n \times n \) symmetric matrix. Then

- \( A \) is positive semidefinite if and only if all the principal minors of \( A \) are nonnegative.
- \( A \) is negative semidefinite if and only if all the \( k \)th order principal minors of \( A \) are \(\leq 0 \) if \( k \) is odd and \(\geq 0 \) if \( k \) is even.

**Example**

Let

\[
A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

The two first-order principal minors and 0 and \(-1\), and the second-order principal minor is 0. Thus the matrix is negative semidefinite. (It is not negative definite, because the first leading principal minor is zero.)

**Procedure for checking the definiteness of a matrix**

- Find the leading principal minors and check if the conditions for positive or negative definiteness are satisfied. If they are, you are done. (If a matrix is positive definite, it is certainly positive semidefinite, and if it is negative definite, it is certainly negative semidefinite.)
- If the conditions are not satisfied, check if they are *strictly* violated. If they are, then the matrix is indefinite.
• If the conditions are not strictly violated, find all its principal minors and check if the conditions for positive or negative semidefiniteness are satisfied.

**Example**

Suppose that the leading principal minors of the $3 \times 3$ matrix $A$ are $D_1 = 1$, $D_2 = 0$, and $D_3 = -1$. Neither the conditions for $A$ to be positive definite nor those for $A$ to be negative definite are satisfied. In fact, both conditions are strictly violated ($D_1$ is positive while $D_3$ is negative), so the matrix is indefinite.

**Example**

Suppose that the leading principal minors of the $3 \times 3$ matrix $A$ are $D_1 = 1$, $D_2 = 0$, and $D_3 = 0$. Neither the conditions for $A$ to be positive definite nor those for $A$ to be negative definite are satisfied. But the condition for positive definiteness is not strictly violated. To check semidefiniteness, we need to examine all the principal minors.

### 2.3 Exercises on quadratic forms

1. Determine whether each of the following quadratic forms in two variables is positive or negative definite or semidefinite, or indefinite.
   a. $x^2 + 2xy$.
   b. $-x^2 + 4xy - 4y^2$.
   c. $-x^2 + 2xy - 3y^2$.
   d. $4x^2 + 8xy + 5y^2$.
   e. $-x^2 + xy - 3y^2$.
   f. $x^2 - 6xy + 9y^2$.
   g. $4x^2 - y^2$.
   h. $(1/2)x^2 - xy + (1/4)y^2$.
   i. $6xy - 9y^2 - x^2$.

2. Determine whether each of the following quadratic forms in three variables is positive or negative definite or semidefinite, or indefinite.
   a. $-x^2 - y^2 - 2z^2 + 2xy$.
   b. $x^2 - 2xy + xz + 2yz + 2z^2 + 3zx$.
   c. $-4x^2 - y^2 + 4xz - 2x^2 + 2y^2$.
   d. $-x^2 - y^2 + 2xz + 4yz + 2z^2$.
   e. $-x^2 + 2xy - 2y^2 + 2xz - 5z^2 + 2yz$.
   f. $y^2 + xy + 2xz$.
   g. $-3x^2 + 2xy - y^2 + 4yz - 8z^2$.
   h. $2x^2 + 2xy + 2y^2 + 4z^2$.

3. Consider the quadratic form $2x^2 + 2xz + 2ayz + 2z^2$, where $a$ is a constant.
   Determine the definiteness of this quadratic form for each possible value of $a$.

4. Determine the values of $a$ for which the quadratic form $x^2 + 2axy + 2xz + z^2$ is positive definite, negative definite, positive semidefinite, negative semidefinite, and indefinite.
5. Consider the matrix
\[
\begin{pmatrix}
  a & 1 & b \\
  1 & -1 & 0 \\
  b & 0 & -2
\end{pmatrix}.
\]

6. Find conditions on \(a\) and \(b\) under which this matrix is negative definite, negative semidefinite, positive definite, positive semidefinite, and indefinite. (There may be no values of \(a\) and \(b\) for which the matrix satisfies some of these conditions.)

7. Show that the matrix
\[
\begin{pmatrix}
  1 & 0 & 1 & 1 \\
  0 & 1 & 0 & 0 \\
  1 & 0 & 1 & 0 \\
  1 & 0 & 0 & 1
\end{pmatrix}
\]
is not positive definite.

### 3.2.3 Solutions to exercises on quadratic forms

1.
   a. The matrix is
\[
\begin{pmatrix}
  1 & 1 \\
  1 & 0
\end{pmatrix}.
\]
   
   b. The determinant is \(-1 < 0\), so the quadratic form is indefinite.
   c. The matrix is
\[
\begin{pmatrix}
  -1 & 2
\end{pmatrix}.
\]
   
   d. The first-order principal minors are \(-1\) and \(-4\); the determinant is 0. Thus the quadratic form is negative semidefinite (but not negative definite, because of the zero determinant).
   e. The matrix is
\[
\begin{pmatrix}
  -1 & 1
\end{pmatrix}.
f. The leading principal minors are $-1$ and $2$, so the quadratic form is negative definite.
g. The associated matrix is
\[
\begin{pmatrix}
4 & 4 \\
4 & 5
\end{pmatrix}
\]
h. The leading principal minors are $4 > 0$ and $(4)(5) - (4)(4) = 4 > 0$. Thus the matrix is positive definite.
i. The associated matrix is
\[
\begin{pmatrix}
-1 & 1/2
\end{pmatrix}
\]
j. The leading principal minors are $-4 < 0$ and $(-1)(-3) - (1/2)(1/2) > 0$. Thus the matrix is negative definite.
k. The associated matrix is
\[
\begin{pmatrix}
1 & -3
\end{pmatrix}
\]
l. The principal minors are $1 > 0$, $9 > 0$, and $(1)(9) - (-3)(-3) = 0$. Thus the matrix is positive semidefinite.
m. The associated matrix is
\[
\begin{pmatrix}
4 & 0
\end{pmatrix}
\]
n. The determinant is $-4 < 0$. Thus the matrix is indefinite.
o. The associated matrix is
\[
\begin{pmatrix}
1/2 & -1/2
\end{pmatrix}
\]
p. The determinant is $(1/2)(1/4) - (-1/2)(-1/2) < 0$. Thus the matrix is indefinite.
q. The associated matrix is
\[
\begin{pmatrix}
-1 & 3
\end{pmatrix}
\]
r. The principal minors are $-1 < 0$, $-9 < 0$, and $(-1)(-9) - (3)(3) = 0$. Thus the matrix is negative semidefinite.

2.

a. The matrix is
\[
\begin{pmatrix}
-1 & 1 & 0 \\
1 & -1 & 0 \\
0 & 0 & -2
\end{pmatrix}.
\]

b. The first-order minors are $-1$, $-1$ and $-2$, the second-order minors are $0$, $2$, and $2$, and the determinant is $0$. Thus the matrix is negative semidefinite.

c. The matrix is
\[
\begin{pmatrix}
1 & -1 & 2 \\
-1 & 0 & 1 \\
2 & 1 & 2
\end{pmatrix}.
\]

d. The first-order principal minors are $1$, $0$, and $2$, so the only possibility is that the quadratic form is positive semidefinite. However, the first second-order principal minor is $-1$. So the matrix is indefinite.

e. The matrix is
\[
\begin{pmatrix}
-4 & 0 & 2 \\
0 & -1 & 1 \\
2 & 1 & -2
\end{pmatrix}.
\]

f. The first-order principal minors are $-4$, $-1$, and $-2$; the second-order principal minors are $4$, $4$, and $1$, and the third-order principal minor is $0$. Thus the matrix is negative semidefinite.

g. The matrix is
\[
\begin{pmatrix}
-1 & 0 & 1 \\
0 & -1 & 2 \\
1 & 2 & 2
\end{pmatrix}.
\]

h. The leading principal minors are $-1$, $1$, and $7$, so the quadratic form is indefinite.

i. The matrix is
The leading principal minors are $-1, 1, \text{ and } 0$, so the matrix is not positive or negative definite, but may be negative semidefinite. The first order principal minors are $-1, -2, \text{ and } -5$; the second-order principal minors are $1, 4, \text{ and } 9$; the third-order principal minor is $0$. Thus the matrix is negative semidefinite.

The matrix is
\[
\begin{pmatrix}
-1 & 1 & 1 \\
1 & -2 & 1 \\
1 & 1 & -5
\end{pmatrix}.
\]

Thus the form is indefinite: one of the first-order principal minors is positive, but the second-order one that is obtained by deleting the third row and column of the matrix is negative.

The matrix is
\[
\begin{pmatrix}
0 & 1/2 & 1 \\
1/2 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

with leading principal minors $-3, 2, \text{ and } -4$. So the form is negative definite.

The matrix is
\[
\begin{pmatrix}
-3 & 1 & 0 \\
1 & -1 & 2 \\
0 & 2 & -8
\end{pmatrix},
\]

The leading principal minors are $2, 3, \text{ and } (2)(8) - (1)(4) = 12 > 0$. Thus the matrix is positive definite.

The matrix is
\[
\begin{pmatrix}
2 & 0 & 1 \\
1 & 2 & 0 \\
0 & 0 & 4
\end{pmatrix}.
\]
4. The first-order minors are 2, 0, and 2, the second-order minors are 0, 3, and \(-a^2\), and determinant \(-2a^2\). Thus for \(a = 0\) the matrix is positive semidefinite, and for other values of \(a\) the matrix is indefinite.

5. The matrix is

\[
\begin{pmatrix}
1 & a & 1 \\
0 & 0 & 0 \\
1 & 0 & 1
\end{pmatrix}.
\]

6. The leading principal minors are 1, \(-a^2\), and \(-a^2\).

7. Thus if \(a \neq 0\) the matrix is indefinite.

8. If \(a = 0\), we need to examine all the principal minors to determine whether the matrix is positive semidefinite. In this case, the first-order principal minors are 1, 0, and 1; the second-order principal minors are 0, 0, and 0; and the third-order principal minor is 0. Thus the quadratic form is positive semidefinite.

9. Conclusion: If \(a \neq 0\) the matrix is indefinite. If \(a = 0\) it is positive semidefinite.

10. The matrix is not positive definite or positive semidefinite for any values of \(a\) and \(b\), because two of the first-order principal minors are negative. Necessary and sufficient conditions for it to be negative definite are
   a. \(a < 0\)
   b. \(-a - 1 > 0\), or \(a < -1\) (looking at first second-order principal minor)
   c. \(2a + 2 + b^2 < 0\) (looking at determinant).

Thus it is negative definite if and only if \(a < -1\) and \(2a + 2 + b^2 < 0\).

It is negative semidefinite if and only if \(a \leq -1\), \(-2a - b^2 \geq 0\), and \(2a + 2 + b^2 \leq 0\). The second condition implies the first, so the matrix is negative semidefinite if and only if \(a \leq -1\) and \(2a + 2 + b^2 \leq 0\).

Otherwise the matrix is indefinite.

o The second order principal minor obtained by deleting the second and fourth rows and columns is 0, so the matrix is not positive definite. (Alternatively, the third-order leading principal minor is 0, and the principal minor obtained by deleting the second and third rows and columns is 0.)

3.3 Concave and convex functions of many variables

Convex sets
To extend the notions of concavity and convexity to functions of many variables we first define the notion of a convex set.
Definition
A set $S$ of $n$-vectors is **convex** if
$$(1−\lambda)x + \lambda x' \in S$$
whenever $x \in S$, $x' \in S$, and $\lambda \in [0,1]$.

We call $(1 − \lambda)x + \lambda x'$ a **convex combination** of $x$ and $x'$.

For example, the two-dimensional set on the left of the following figure is convex, because the line segment joining every pair of points in the set lies entirely in the set. The set on the right is not convex, because the line segment joining the points $x$ and $x'$ does not lie entirely in the set.

![Diagram showing convex and non-convex sets]

The following property of convex sets (which you are asked to prove in an exercise) is sometimes useful.

**Proposition**
The intersection of convex sets is convex.

Note that the union of convex sets is not necessarily convex.

**Convex and concave functions**
Let $f$ be a function of many variables, defined on a convex set $S$. We say that $f$ is **concave** if the line segment joining any two points on the graph of $f$ is never above the graph; $f$ is **convex** if the line segment joining any two points on the graph is never below the graph. (That is, the definitions are the same as the definitions for functions of a single variable.)

More precisely, we can make the following definition (which is again essentially the same as the corresponding definition for a function of a single variable). Note that only functions defined on convex sets are covered by the definition.

**Definition**
Let $f$ be a function of many variables defined on the convex set $S$. Then $f$ is
- **concave** on the set $S$ if for all $x \in S$, all $x' \in S$, and all $\lambda \in (0,1)$ we have
  $$f((1−\lambda)x + \lambda x') \geq (1−\lambda)f(x) + \lambda f(x')$$
- **convex** on the set $S$ if for all $x \in S$, all $x' \in S$, and all $\lambda \in (0,1)$ we have...
\[ f((1-\lambda)x + \lambda x') \leq (1-\lambda)f(x) + \lambda f(x'). \]

**Once again,** a **strictly concave** function is one that satisfies the definition for concavity with a strict inequality (> rather than \(\geq\)) for all \(x \neq x'\), and a **strictly convex** function is one that satisfies the definition for concavity with a strict inequality (< rather than \(\leq\)) for all \(x \neq x'\).

**Example**

Let \( f \) be a linear function, defined by \( f(x) = a_1x_1 + \ldots + a_nx_n = a \cdot x \) on a convex set, where \( a \) is a constant for each \( i \). Then \( f \) is both concave and convex:

\[
\begin{align*}
  f((1-\lambda)x + \lambda x') &= a \cdot [(1-\lambda)x + \lambda x'] \\
    &= (1-\lambda)a \cdot x + \lambda a \cdot x' \\
    &= (1-\lambda)f(x) + \lambda f(x')
\end{align*}
\]

for all \( x, x', \) and \( \lambda \in [0, 1] \).

**Example**

Suppose the function \( g \) of a single variable is concave on \([a, b]\), and the function \( f \) of two variables is defined by \( f(x, y) = g(x) \) on \([a, b] \times [c, d] \). Is \( f \) concave?

First note that the domain of \( f \) is a convex set, so the definition of concavity can apply.

The functions \( g \) and \( f \) are illustrated in the following figure. (The axes for \( g \) are shown in perspective, like those for \( f \), to make the relation between the two figures clear. If we were plotting only \( g \), we would view it straight on, so that the \( x \)-axis would be horizontal. Note that every cross-section of the graph of \( f \) parallel to the \( x \)-axis is the graph of the function \( g \).)

From the graph of \( f \) (the roof of a horizontal tunnel), you can see that it is concave. The following argument is precise:

\[
\begin{align*}
  f((1-\lambda)(x, y) + \lambda(x', y')) &= f((1-\lambda)x + \lambda x', (1-\lambda)y + \lambda y') \\
    &= g((1-\lambda)x + \lambda x')
\end{align*}
\]
\[ \geq (1-\lambda)g(x) + \lambda g(x') \]
\[ = (1-\lambda) f(x, y) + \lambda f(x', y') \]

so \( f \) is concave.

**Example**

Let \( f \) and \( g \) be defined as in the previous example. Assume now that \( g \) is **strictly** concave. Is \( f \) strictly concave?

The strict concavity of \( f \) implies that

\[ f \left((1-\lambda)(x, y) + \lambda(x', y')\right) > (1-\lambda) f(x, y) + \lambda f(x', y') \]

for all \( x \neq x' \). But to show that \( f \) is strictly concave we need to show that the inequality is strict whenever \( (x, y) \neq (x', y') \)--in particular, for cases in which \( x = x' \) and \( y \neq y' \). In such a case, we have

\[ f \left((1-\lambda)(x, y) + \lambda(x', y')\right) = f(x, (1-\lambda)y + \lambda y') = g(x) = (1-\lambda) f(x, y) + \lambda f(x, y'). \]

Thus \( f \) is not strictly concave. You can see the lack of strict concavity in the figure (in the previous example): if you take two \((x, y)\) pairs with the same value of \( x \), the line joining them lies everywhere on the surface of the function, never below it.

**A characterization of convex and concave functions**

Having seen many examples of concave functions, you should find it plausible that a function is concave if and only if the set of points under its graph---the set shaded pink in the following figure---is convex. The result is stated precisely in the following proposition.

![Graph of f(x) with shaded region]

**Proposition**

A function \( f \) of many variables defined on the convex set \( S \) is
• concave if and only if the set of points below its graph is convex:
  \{(x, y): x \in S \text{ and } y \leq f(x)\} \text{ is convex}

• convex if and only if the set of points above its graph is convex:
  \{(x, y): x \in S \text{ and } y \geq f(x)\} \text{ is convex.}

**How can we tell if a twice-differentiable function is concave or convex?**

A twice-differentiable function of a single variable is concave if and only if its second derivative is nonpositive everywhere.

To determine whether a twice-differentiable function of many variables is concave or convex, we need to examine all its second partial derivatives. We call the matrix of all the second partial derivatives the **Hessian** of the function.

**Definition**

Let \( f \) be a twice differentiable function of \( n \) variables. The **Hessian** of \( f \) at \( x \) is

\[
H(x) = \begin{bmatrix}
  f_{11}''(x) & f_{12}''(x) & \cdots & f_{1n}''(x) \\
  f_{21}''(x) & f_{22}''(x) & \cdots & f_{2n}''(x) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{n1}''(x) & f_{n2}''(x) & \cdots & f_{nn}''(x)
\end{bmatrix}.
\]

Note that by **Young's theorem**, the Hessian of any function for which all second partial derivatives are continuous is symmetric for all values of the argument of the function.

We can determine the concavity/convexity of a function by determining whether the Hessian is negative or positive semidefinite, as follows.

**Proposition**

Let \( f \) be a function of many variables with continuous partial derivatives of first and second order on the convex open set \( S \) and denote the Hessian of \( f \) at the point \( x \) by \( H(x) \). Then

- \( f \) is concave if and only if \( H(x) \) is negative semidefinite for all \( x \in S \)
- if \( H(x) \) is negative definite for all \( x \in S \) then \( f \) is strictly concave
- \( f \) is convex if and only if \( H(x) \) is positive semidefinite for all \( x \in S \)
- if \( H(x) \) is positive definite for all \( x \in S \) then \( f \) is strictly convex.
Note that the result does not claim that if \( f \) is strictly concave then \( H(x) \) is negative definite for all \( x \in S \). Indeed, consider the function \( f \) of a single variable defined by \( f(x) = -x^4 \). This function is strictly concave, but the \( 1 \times 1 \) matrix \( H(0) \) is not negative definite (its single component is 0).

Thus if you want to determine whether a function is strictly concave or strictly convex, you should first check the Hessian. If the Hessian is negative definite for all values of \( x \) then the function is strictly concave, and if the Hessian is positive definite for all values of \( x \) then the function is strictly convex. If the Hessian is not negative semidefinite for all values of \( x \) then the function is not concave, and hence of course is not strictly concave. Similarly, if the Hessian is not positive semidefinite the function is not convex. If the Hessian is not negative definite for all values of \( x \) but is negative semidefinite for all values of \( x \), the function may or may not be strictly concave; you need to use the basic definition of strict concavity to determine whether it is strictly concave or not. Similarly, if the Hessian is not positive definite for all values of \( x \) but is positive semidefinite for all values of \( x \), the function may or may not be strictly convex.

**Example**

Consider the function \( f(x, y) = 2x - y - x^2 + 2xy - y^2 \). Its Hessian is

\[
\begin{bmatrix}
-2 & 2 \\
2 & 2
\end{bmatrix}
\]

which is negative semidefinite. (In this case the Hessian does not depend on \((x, y)\); in general it does.) Thus \( f \) is concave.

**Example**

Consider the function \( f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 \). Its first partials are

\[
\begin{align*}
\frac{\partial f}{\partial x_1} &= 2x_1 + 2x_2 + 2x_3 \\
\frac{\partial f}{\partial x_2} &= 4x_2 + 2x_1 \\
\frac{\partial f}{\partial x_3} &= 6x_3 + 2x_1.
\end{align*}
\]

So its Hessian is

\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 4 & 0 \\
2 & 0 & 6
\end{bmatrix}
\]

The leading principal minors of the Hessian are \( 2 > 0 \), \( 4 > 0 \), and \( 8 > 0 \). So the Hessian is positive definite, and \( f \) is strictly convex.

In these two examples, the Hessian of \( f \) is independent of its argument, because \( f \) is a quadratic. In the next example, the Hessian of the function does not have this property.
Consider the Cobb-Douglas function, defined by \( f(K, L) = AK^aL^b \). The Hessian of this function is

\[
\begin{pmatrix}
  a(a-1)AK^{a-2}L^b & abAK^{a-1}L^{b-1}
\end{pmatrix}
\]

Thus in order that \( f \) be concave we need \( a(a-1)AK^{a-2}L^b \leq 0 \), \( b(b-1)AK^{a-2}L^b \leq 0 \), and \( abAK^{a-1}L^{b-1}(1-(a+b)) \geq 0 \). Thus \( f \) is concave if \( A \geq 0 \), \( a \geq 0 \), \( b \geq 0 \), and \( a + b \leq 1 \), and is strictly concave if \( A > 0 \), \( a > 0 \), \( b > 0 \), and \( a + b < 1 \).

If we have a function that is a sum of functions that we know are concave, or is a concave increasing function of a concave function, the following result is useful. The last two parts of this result generalize to functions of many variables a previous result. (The proof is the same as the proof for functions of a single variable.)

**Proposition**

- If the functions \( f \) and \( g \) are concave and \( a \geq 0 \) and \( b \geq 0 \) then the function \( af + bg \) is concave.
- If the functions \( f \) and \( g \) are convex and \( a \geq 0 \) and \( b \geq 0 \) then the function \( af + bg \) is convex.
- If the function \( U \) is concave and the function \( g \) is nondecreasing and concave then the function \( f \) defined by \( f(x) = g(U(x)) \) is concave.
- If the function \( U \) is convex and the function \( g \) is nondecreasing and convex then the function \( f \) defined by \( f(x) = g(U(x)) \) is convex.

**Example**

A firm produces the output \( f(x) \) from the vector \( x \) of inputs, which costs it \( c(x) \). The function \( f \) is concave and the function \( c \) is convex. The firm sells its output at a fixed price \( p > 0 \). Its profit when it uses the input vector \( x \) is

\[
\pi(x) = pf(x) - c(x).
\]

That is, \( \pi \) is the sum of two functions, \( pf \) and \( -c \). The function \( -c \) is concave because \( c \) is convex, so by the proposition \( \pi \) is concave.

### 3.3 Exercises on convexity and concavity for functions of many variables

1. By drawing diagrams, determine which of the following sets is convex.
   a. \( \{(x, y): y = e^x\} \).
   b. \( \{(x, y): y \geq e^x\} \).
   c. \( \{(x, y): xy \geq 1, x > 0, y > 0\} \).
2. Show that the intersection of two convex sets is convex.
3. For each of the following functions, determine which, if any, of the following conditions the function satisfies: concavity, strict concavity, convexity, strict convexity. (Use whatever technique is most appropriate for each case.)
   a. \( f(x, y) = x + y \).
   b. \( f(x, y) = x^2 \). [Note: \( f \) is a function of two variables.]
   c. \( f(x, y) = x + y - e^x - e^y \).
   d. \( f(x, y, z) = x^2 + y^2 + 3z^2 - xy + 2xz + yz \).

4. The function \( C \) of many variables and the function \( D \) of a single variable are both convex. Define the function \( f \) by \( f(x, k) = C(x) + D(k) \). Show that \( f \) is a convex function (without assuming that \( C \) and \( D \) are differentiable).

5. The function \( f \) (of \( n \) variables) is concave, and the function \( g \) (of \( n \) variables) is convex. Neither function is necessarily differentiable. Is the function \( h \) defined by \( h(x) = af(x) - bg(x) \), where \( a \geq 0 \) and \( b \geq 0 \) are constants, necessarily concave? (Either show it is, or show it isn't.) Your argument should use only the definition of concavity, and should not refer to any result mentioned in class (or in the book). You won't get any credit for an argument that assumes that \( f \) and \( g \) are differentiable.

6. The functions \( f \), of many variables, and \( g \), of a single variable, are concave, but not necessarily differentiable. Define the function \( h \) by \( h(x) = g(f(x)) \) for all \( x \). Is the function \( h \) necessarily concave? Necessarily not concave? (Give a complete argument; you will get no credit for an argument that applies only if \( f \) and \( g \) are differentiable.)

7. Let \( f(x_1, x_2) = x_1^2 - x_1x_2 + x_2^2 + 3x_1 - 2x_2 + 1 \). Is \( f \) convex, concave, or neither?

8. Suppose that a firm that uses 2 inputs has the production function \( f(x_1, x_2) = 12x_1^{1/3}x_2^{1/2} \) and faces the input prices \((p_1, p_2)\) and the output price \( q \). Show that \( f \) is concave for \( x_1 > 0 \) and \( x_2 > 0 \), so that the firm's profit is concave. (Show that \( f \) is concave from first principles, not from the general result about the concavity of Cobb-Douglas production functions.)

9. Let \( f(x_1, x_2) = x_1^2 + 2x_2^2 + 2x_1x_2 + (1/2)x_2^2 - 8x_1 - 2x_2 - 8 \). Find the range of values of \((x_1, x_2)\) for which \( f \) is convex, if any.

10. Determine the values of \( a \) (if any) for which the function \( 2x^2 + 2xz + 2ayz + 2z^2 \) is concave and the values for which it is convex.

Show that the function \(-w^2 + 2wx - x^2 - y^2 + 4yz - z^2\) (in the four variables \(w, x, y, z\)) 3.3 Solutions to exercises on convexity and concavity for functions of many variables

11. a. Not convex, because \( e^{w(x_1 - x_2)} \neq \theta e^x + (1-\theta)e^y \), as illustrated in the following figure.
b. Convex, because $e^{\theta x + (1 - \theta)u} < \theta e^x + (1 - \theta)e^u$ (see the following figure).

c. Convex, because if $xy \geq 1$ and $uv \geq 1$ then $(\theta x + (1 - \theta)u)(\theta y + (1 - \theta)v) \geq 1$ (see figure).

12. Let $A$ and $B$ be convex sets. Let $x \in A \cap B$ and $x' \in A \cap B$. We need to show that $(1 - \lambda)x + \lambda x' \in A \cap B$ for all $\lambda \in [0, 1]$. Since $x \in A$, $x' \in A$, and $A$ is convex we have $(1 - \lambda)x + \lambda x' \in A$ for all $\lambda \in [0, 1]$. Similarly $(1 - \lambda)y + \lambda y' \in B$ for all $\lambda \in [0, 1]$. Hence $(1 - \lambda)x + \lambda x' \in A \cap B$ for all $\lambda \in [0, 1]$.

13. a. $f((1 - \lambda)x + \lambda x', (1 - \lambda)y + \lambda y') = (1 - \lambda)f(x, y) + \lambda f(x', y')$, so the function is concave and convex, but not strictly concave or strictly convex. Or you can calculate the Hessian, from
which you can conclude that the function is both concave and convex, and then argue as above that the function is not strictly concave or strictly convex. [Note: the fact that some of the minors are zero does not imply that the function is not strictly concave or strictly convex, although in fact it is not.] Or you can appeal to the fact that the function is linear to conclude that it is concave and convex.

b. The Hessian shows that the function is convex (all principal minors are nonnegative). The Hessian does not satisfy the sufficient condition for strict convexity, but this does not imply that the function is in fact not strictly convex. However, since, for example, \( f(1, 1) = f(1, 2) = f(1, 3) \), we have \[ f((1-\lambda)(x,y) + \lambda(x',y')) = f(x,(1-\lambda)y + \lambda y') = x^2 = (1-\lambda) f(x,y) + \lambda f(x,y'). \]

c. The Hessian is
\[
\begin{pmatrix}
-e^x & -e^x \\
-e^x & e^x
\end{pmatrix}
\]

d. Since \(-e^x - e^x < 0\) for all \((x,y)\) and the determinant is
\[ (-e^x - e^x)(-e^x) - (-e^x)(-e^x) = e^{2x} > 0 \] for all \((x,y)\) the function is strictly concave. (Or you can argue that since \(e^x\) is increasing and convex and \(x + y\) is convex, \(e^x\) is convex and thus \(-e^x\) is concave, and similarly for \(-e^x\); then you need to make a separate argument for strict concavity.)

e. The Hessian is
\[
\begin{pmatrix}
2 & -1 & 2 \\
-1 & 2 & 1 \\
2 & 1 & 6
\end{pmatrix}
\]
f. the leading principal minors of which are 2 > 0, 3 > 0, and 4 > 0, so that the function is strictly convex.

14. We have
\[
f(((1-\lambda)(x,k) + \lambda(x',k')) = C((1-\lambda)x + \lambda x') + D((1-\lambda)k + \lambda k') \\
\leq (1-\lambda)C(x) + \lambda C(x') + (1-\lambda)D(k) + \lambda D(k')
\]

15. by the convexity of \(C\) and \(D\). But the right-hand side is \((1-\lambda) f(x,k) + \lambda f(x',k')\), showing that \(f\) is convex.

16. The function \(f\) is concave, so
\[
f((1-\alpha)x + \alpha y) \geq (1 - \alpha) f(x) + \alpha f(y) \quad \text{for all } x, y, \text{ and } \alpha \in [0, 1]
\] and the function \(g\) is convex, so
\[
g((1-\alpha)x + \alpha y) \leq (1 - \alpha)g(x) + \alpha g(y) \quad \text{for all } x, y, \text{ and } \alpha \in [0, 1].
\]
Thus
\[ h((1-\alpha)x + \alpha y) = a f((1-\alpha)x + \alpha y) - b g((1-\alpha)x + \alpha y) \]
\[ \geq a(1-\alpha) f(x) + a\alpha f(y) - b(1-\alpha)g(x) - b\alpha g(y). \]

Now, the last expression is equal to \((1-\alpha)(a f(x) - b g(x)) + \alpha(a f(y) - b g(y))\),
which is equal to \((1-\alpha)h(x) + \alpha h(y)\), so that \(h\) is concave.

17. The function \(h\) is not necessarily concave: if, for example, \(f(x) = -x^2\) and \(g(z) = -z\) (which are both concave), then \(h(x) = x^2\), which is not concave (it is strictly convex).

The function \(h\) is also not necessarily not concave: if, for example, \(f(x) = x\) and \(g(z) = z\) (which are both concave), then \(h(x) = x\), which is concave.

18. The Hessian matrix of \(f\) is
\[
\begin{pmatrix}
2 & -1 \\
-1 & 1
\end{pmatrix}
\]

19. This matrix is positive definite, so \(f\) is convex.

20. The Hessian matrix at \((x_1, x_2)\) is
\[
\begin{pmatrix}
-(8/3)x_1^{-3/2}x_2^{1/2} & 2x_1^{-1/2}x_2^{-1/2} \\
2x_1^{-1/2}x_2^{-1/2} & 1
\end{pmatrix}
\]

21. The leading principal minors are \(-(8/3)x_1^{-3/2}x_2^{1/2} < 0\) and \(8x_1^{-3/2}x_2^{-1} - 4x_1^{-1/2}x_2^{-1} = 4x_1^{-1/2}x_2^{-1} > 0\) for \(x_1 > 0\) and \(x_2 > 0\). Hence the Hessian is negative definite, so that \(f\) is concave.

22. The Hessian matrix of \(f\) is
\[
\begin{pmatrix}
6x_1 + 4 & 2 \\
2 & 1
\end{pmatrix}
\]

23. This matrix is positive semidefinite if \(6x_1 + 4 \geq 0\) and \(6x_1 \geq 0\), or if \(x_1 \geq 0\). Thus \(f\) is convex for \(x_1 \geq 0\) (and all \(x_i\)).

24. The Hessian of the function is
\[
\begin{pmatrix}
4 & 0 & 2 \\
0 & 0 & 2a \\
2 & 2a & 4
\end{pmatrix}
\]
25. The first-order minors are 4, 0, and 4, the second-order minors are 0, 6, and \(-2a^2\), and determinant \(-4a^2\). Thus for \(a = 0\) the Hessian is positive semidefinite, so that the function is convex; for other values of \(a\) the Hessian is indefinite, so that the function is neither concave nor convex.

26. The Hessian is

\[
\begin{pmatrix}
-2 & 2 & 0 & 0 \\
2 & -2 & 0 & 0 \\
0 & 0 & -2 & 4 \\
0 & 0 & 4 & -2
\end{pmatrix}
\]

27. The second order principal minor obtained by deleting the first and second rows and columns is \(-12\), so the matrix is not negative semidefinite. Thus the function is not concave.

3.4 Quasiconcavity and quasiconvexity

**Definitions and basic properties**

Think of a mountain in the Swiss Alps: cows grazing on the verdant lower slopes, snow capping the majestic peak.

Now forget about the cows and the snow. Ask yourself whether the function defining the surface of the mountain is concave. It is if every straight line connecting two points on the surface lies everywhere on or under the surface.

If, for example, the mountain is a perfect dome (half of a sphere), then this condition is satisfied, so that the function defined by its surface is concave. The condition is satisfied also if the mountain is a perfect cone. In this case, every straight line connecting two points on the surface lies exactly on the surface.

Now suppose that the mountain is a deformation of a cone that gets progressively steeper at higher altitudes---call it a "pinched cone". (Many mountains seem to have this characteristic when you try to climb them!) That is, suppose that when viewed from far away, the mountain looks like this:
In this case, a straight line from the top of the mountain to any other point on the surface does not lie on or under the surface, but rather passes through clear air. Thus the function defined by the surface of the mountain is not concave.

The function does, however, share a property with a perfect dome and a cone: on a topographic map of the mountain, the set of points inside each contour---the set of points at which the height of the mountain exceeds any given number---is convex. In fact, each contour line of this mountain, like each contour line of a perfect dome and of a cone, is a circle. If we draw contour lines for regularly-spaced heights (e.g. 50m, 100m, 150m, ...), then topographic maps of the three mountains look like this:

The spacing of the contour lines differs, but the set of points inside every contour has the same shape for each mountain---it is a disk. In particular, every such set is convex.

If we model the surface of the mountain as a function \( f \) of its longitude and latitude \((x, y)\), then a contour is a level curve of \( f \). A function with the property that for every value of \( a \) the set of points \((x, y)\) such that \( f(x, y) \geq a \)---the set of points inside every contour on a topographic map---is convex is said to be quasiconcave.

Not every mountain has this property. In fact, if you take a look at a few maps, you'll see that almost no mountain does. A topographic map of an actual mountain is likely to look something like this:
The three outer contours of this mountain definitely do not enclose convex sets. Take, for example, the one in red. The blue line, connecting two points in the set enclosed by the contour, goes outside the set.

Thus the function defined by the surface of this mountain is not quasiconcave.

Let $f$ be a multivariate function defined on the set $S$. We say that $f$ (like the function defining the surface of a mountain) is quasiconcave if, for any number $a$, the set of points for which $f(x) \geq a$ is convex. For any real number $a$, the set

$$P_a = \{ x \in S : f(x) \geq a \}$$

is called the upper level set of $f$ for $a$. (In the case of a mountain, $P_a$ is the set of all points at which the altitude is at least $a$.)

**Example**

Let $f(x, y) = -x^2 - y^2$. The upper level set of $f$ for $a$ is the set of pairs $(x, y)$ such that $-x^2 - y^2 \geq a$, or $x^2 + y^2 \leq -a$. Thus for $a > 0$ the upper level set $P_a$ is empty, and for $a < 0$ it is a disk of radius $a$.

**Definition**

The multivariate function $f$ defined on a convex set $S$ is quasiconcave if every upper level set of $f$ is convex. (That is, $P_a = \{ x \in S : f(x) \geq a \}$ is convex for every value of $a$.)

The notion of quasiconvexity is defined analogously. First, for any real number $a$, the set

$$P_a = \{ x \in S : f(x) \leq a \}$$

is the set of all the points that yield a value for the function of at most $a$; it is called the lower level set of $f$ for $a$. (In the case of a mountain, $P_a$ is the set of all points at which the altitude is at most $a$.)

**Definition**
The multivariate function $f$ defined on a convex set $S$ is **quasiconvex** if every lower level set of $f$ is convex. (That is, $P_a = \{x \in S : f(x) \leq a\}$ is convex for every value of $a$.)

Note that $f$ is quasiconvex if and only if $-f$ is quasiconcave.

The notion of quasiconcavity is weaker than the notion of concavity, in the sense that every concave function is quasiconcave. Similarly, every convex function is quasiconvex.

**Proposition**

A concave function is quasiconcave. A convex function is quasiconvex.

**Proof**

Denote the function by $f$, and the (convex) set on which it is defined by $S$. Let $a$ be a real number and let $x$ and $y$ be points in the upper level set $P_a$: $x \in P_a$ and $y \in P_a$. We need to show that $P_a$ is convex. That is, we need to show that for every $\lambda \in [0,1]$ we have $(1-\lambda)x + \lambda y \in P_a$.

First note that the set $S$ on which $f$ is defined is convex, so we have $(1-\lambda)x + \lambda y \in S$, and thus $f$ is defined at the point $(1-\lambda)x + \lambda y$.

Now, the concavity of $f$ implies that

$$f((1-\lambda)x + \lambda y) \geq (1-\lambda)f(x) + \lambda f(y).$$

Further, the fact that $x \in P_a$ means that $f(x) \geq a$, and the fact that $y \in P_a$ means that $f(y) \geq a$, so that

$$(1-\lambda)f(x) + \lambda f(y) \geq (1-\lambda)a + \lambda a = a.$$

Combining the last two inequalities, we have

$$f((1-\lambda)x + \lambda y) \geq a,$$

so that $(1-\lambda)x + \lambda y \in P_a$. Thus every upper level set is convex and hence $f$ is quasiconcave.

The converse of this result is not true: a quasiconcave function may not be concave. Consider, for example, the function $f(x, y) = xy$ defined on the set of pairs of nonnegative real numbers. This function is quasiconcave (its upper level sets are the sets of points above rectangular hyperbolae), but is not concave (for example, $f(0, 0) = 0$, $f(1, 1) = 1$, and $f(2, 2) = 4$, so that $f((1/2)(0, 0) + (1/2)(2, 2)) = f(1, 1) = 1 < 2 = (1/2)f(0, 0) + (1/2)f(2, 2)$).

Some properties of quasiconcave functions are given in the following result. (You are asked to prove the first result in an exercise.)

**Proposition**

- If the function $U$ is quasiconcave and the function $g$ is increasing then the
function \( f \) defined by \( f(x) = g(U(x)) \) is quasiconcave.

- If the function \( U \) is quasiconcave and the function \( g \) is decreasing then the function \( f \) defined by \( f(x) = g(U(x)) \) is quasiconvex.

In an exercise you are asked to show that the sum of quasiconcave functions may not be quasiconcave.

**Why are economists interested in quasiconcavity?**

The standard model of a decision-maker in economic theory consists of a set of alternatives and an ordering over these alternatives. The decision-maker is assumed to choose her favorite alternative---that is, an alternative with the property that no other alternative is higher in her ordering.

To facilitate the analysis of such a problem, we often work with a function that "represents" the ordering. Suppose, for example, that there are four alternatives, \( a, b, c, \) and \( d \), and the decision-maker prefers \( a \) to \( b \) to \( c \) and regards \( c \) and \( d \) as equally desirable. This ordering is represented by the function \( U \) defined by \( U(a) = 3, U(b) = 2, \) and \( U(c) = U(d) = 1 \). It is represented also by many other functions---for example \( V \) defined by \( V(a) = 100, V(b) = 0, \) and \( V(c) = V(d) = -1 \). The numbers we assign to the alternatives are unimportant except insofar as they are ordered in the same way that the decision-maker orders the alternatives. Thus any function \( W \) with \( W(a) > W(b) > W(c) = W(d) \) represents the ordering.

When the decision-maker is a consumer choosing between bundles of goods, we often assume that the level curves of the consumer's ordering---which we call "indifference curves"---look like this

That is, we assume that every upper level set of the consumer's ordering is convex, which is equivalent to the condition that any function that represents the consumer's ordering is quasiconcave.

It makes no sense to impose a stronger condition, like concavity, on this function, because the only significant property of the function is the character of its level curves, not the specific numbers assigned to these curves.
Functions of a single variable

The definitions above apply to any function, including those of a single variable. For a function of a single variable, an upper or lower level set is typically an interval of points, or a union of intervals. In the following figure, for example, the upper level set for the indicated value \( a \)--that is, the set of values of \( x \) for which \( f(x) \geq a \)--is the union of the two blue intervals of values of \( x \): the set of all values that are either between \( x' \) and \( x'' \) or greater than \( x''' \).

By drawing some examples, you should be able to convince yourself of the following result.

**Proposition**

A function \( f \) of a single variable is quasiconcave if and only if either

- it is nondecreasing,
- it is nonincreasing, or
- there exists \( x^* \) such that \( f \) is nondecreasing for \( x < x^* \) and nonincreasing for \( x > x^* \).

Note that this result does **NOT** apply to functions of many variables!

**Another characterization of quasiconcavity**

The following alternative characterization of a quasiconcave function (of any number of variables) is sometimes useful.

**Proposition**

The function \( f \) is quasiconcave if and only if for all \( x \in S \), all \( x' \in S \), and all \( \lambda \in [0,1] \) we have if \( f(x) \geq f(x') \) then \( f((1-\lambda)x + \lambda x') \geq f(x') \).

That is, a function is quasiconcave if and only if the line segment joining the points on two level curves lies nowhere below the level curve corresponding to the lower value of the function. This condition is illustrated in the following figure, in which \( a' > a \): all points on the green line, joining \( x \) and \( x' \), lie on or above the indifference curve corresponding to the smaller value of the function \( a \).
Strict quasiconcavity

This characterization of quasiconcavity motivates the following definition of a strictly quasiconcave function.

**Definition**

The multivariate function $f$ defined on a convex set $S$ is **strictly quasiconcave** if for all $x \in S$, all $x' \in S$ with $x' \neq x$, and all $\lambda \in (0, 1)$ we have

\[
\text{if } f(x) \geq f(x') \text{ then } f((1-\lambda)x + \lambda x') > f(x').
\]

That is, a function is strictly quasiconcave if every point, except the endpoints, on any line segment joining points on two level curves yields a higher value for the function than does any point on the level curve corresponding to the lower value of the function.

The definition says that a quasiconcave function of a single variable is strictly quasiconcave if its graph has no horizontal sections.

For a function of two variables, it says that no level curve of a strictly quasiconcave function contains a line segment. (Take $x = x'$ in the definition.) Two examples of functions that are not strictly quasiconcave, though the level curves indicated are consistent with the functions' being quasiconcave, are shown in the following figure. In both cases, the red level curve contains a line segment. (In the diagram on the right, it does so because it is "thick"---see the earlier example.)

**How can we tell if a twice-differentiable function is quasiconcave or quasiconvex?**

To determine whether a twice-differentiable function is quasiconcave or quasiconvex, we can examine the determinants of the **bordered Hessians** of the function, defined as follows:
Notice that a function of $n$ variables has $n$ bordered Hessians, $D_1, \ldots, D_n$.

Proposition

Let $f$ be a function of $n$ variables with continuous partial derivatives of first and second order in an open convex set $S$.

- If $f$ is quasiconcave then $D_1(x) \leq 0$, $D_2(x) \geq 0$, ..., $D_n(x) \leq 0$ if $n$ is odd and $D_n(x) \geq 0$ if $n$ is even, for all $x$ in $S$. (Note that the first condition is automatically satisfied.)
- If $f$ is quasiconvex then $D_k(x) \leq 0$ for all $k$, for all $x$ in $S$. (Note that the first condition is automatically satisfied.)
- If $D_1(x) < 0$, $D_2(x) > 0$, ..., $D_n(x) < 0$ if $n$ is odd and $D_n(x) > 0$ if $n$ is even for all $x$ in $S$ then $f$ is quasiconcave.
- If $D_1(x) < 0$ for all $k$, for all $x$ in $S$ then $f$ is quasiconvex.

Another way to state this result is to say that "$D_1(x) \leq 0$, $D_2(x) \geq 0$, ..., $D_n(x) \leq 0$ if $n$ is odd and $D_n(x) \geq 0$ if $n$ is even, for all $x$ in $S$" is a necessary condition for quasiconcavity, whereas "$D_1(x) < 0$, $D_2(x) > 0$, ..., $D_n(x) < 0$ if $n$ is odd and $D_n(x) > 0$ if $n$ is even for all $x$ in $S$" is a sufficient condition, and similarly for quasiconvexity.

Note that the conditions don't cover all the possible cases, unlike the analogous result for concave functions. If, for example, $D_k(x) \leq 0$ for all $k$, for all $x$, but $D_r(x) = 0$ for some $r$ and some $x$, then the result does not rule out the possibility that the function is quasiconvex, but it does not tell us that it is.

Example

Consider the function $f(x) = x^2$ for $x > 0$. We have $D_1(x) = -4x < 0$ for all $x > 0$, so we deduce that this function is both quasiconcave and quasiconvex on the set $\{x: x > 0\}$.

Example

Consider the function $f(x) = x^2$ for $x \geq 0$. We have $D_1(0) = 0$, so this function does not satisfy the sufficient conditions for either quasiconcavity or quasiconvexity, although it is in fact both quasiconcave and quasiconvex.
Consider the function $f(x_1, x_2) = x_1 x_2$. For $x > 0$ the sufficient conditions for quasiconcavity are satisfied, while the necessary conditions for quasiconvexity are not. Thus the function is quasiconcave and not quasiconvex on the set $\{x: x > 0\}$. For $x \geq 0$ the sufficient conditions for quasiconcavity are not satisfied, but the necessary conditions are not violated. (The function is in fact quasiconcave on this domain.)

### 3.4 Exercises on quasiconcavity and quasiconvexity

1. Draw the upper level sets of each of the following functions for the indicated values. In each case, say whether the set is consistent with the function's being quasiconcave. (Of course, the shape of one upper level set does not determine whether the function is quasiconcave, which is a property of all the upper level sets; the question is only whether the single upper level set your draw is consistent with quasiconcavity.)
   a. $f(x, y) = xy$ for the value 1.
   b. $f(x, y) = x^2 + y^2$ for the value 1.
   c. $f(x, y) = -x^2 - y^2$ for the value -1.

2. For what values of the parameters $a$, $b$, $c$, and $d$ is the function $ax^3 + bx^2 + cx + d$ quasiconvex? (Use the characterization of quasiconcave functions of a single variable. There are several cases to work through.)

3. Use the bordered Hessian condition to determine whether the function $f(x, y) = ye^{x-1}$ is quasiconcave for the region in which $x \geq 0$ and $y \geq 0$.

4. Give an example to show that the sum of quasiconcave functions is not necessarily quasiconcave.

5. The function $f$ is concave and the function $g$ is quasiconcave; neither is necessarily differentiable. Is the function $h$ defined by $h(x) = f(x) + g(x)$ necessarily quasiconcave? (Either show it is, or show it isn't.)

6. The functions $f$ and $g$ of a single variable are concave (but not necessarily differentiable). Is the function $h$ defined by $h(x) = f(x)g(x)$ necessarily quasiconcave?

7. Determine, if possible, which of the following properties each of the following functions satisfies: convexity, strict convexity, concavity, strict concavity, quasiconvexity, quasiconcavity, strict quasiconvexity, strict quasiconcavity.
   a. $f(x, y) = x^2y^2$ for $x \geq 0$ and $y \geq 0$.
   b. $f(x, y) = x - e^x - e^{-y}$.

8. The function $f$ of two variables is defined by $f(x_1, x_2) = x_1(x_2)^2$. If possible, determine whether this function is quasiconcave for $x_1 > 0$ and $x_2 > 0$. If not possible, say why not.

9. The three curves in the figure below are the sets of points for which the value of the function $f$ (of two variables, $x$ and $y$) is equal to 1, 2, and 4. Are these curves consistent or inconsistent with the function's being
   a. quasiconcave?
   b. concave?
10. Show that a concave function is quasiconcave by using the fact that a function \( f \) is quasiconcave if and only if for all \( x \in S \), all \( y \in S \), and all \( \lambda \in [0,1] \) we have

\[
\text{if } f(x) \geq f(y) \text{ then } f((1-\lambda)x + \lambda y) \geq f(y).
\]

11. Reminder: A function \( f \) is quasiconcave if and only if for every \( x \) and \( y \) and every \( \lambda \) with \( 0 \leq \lambda \leq 1 \), if \( f(x) \geq f(y) \) then \( f((1-\lambda)x + \lambda y) \geq f(y) \).

Suppose that the function \( U \) is quasiconcave and the function \( g \) is increasing. Show that the function \( f \) defined by \( f(x) = g(U(x)) \) is quasiconcave.

**3.4 Solutions to exercises on quasiconcavity and quasiconvexity**

1.

a. The set is illustrated in red in the following figure. It is not convex, and hence is not consistent with the function's being quasiconcave. (If the domain of the function were restricted to \( x \geq 0 \) and \( y \geq 0 \), then it would be quasiconcave.)
b. The set is illustrated in red in the following figure. It is not convex, and hence not consistent with the function's being quasiconcave. (That is, the function is definitely not quasiconcave.)

![Diagram of a non-convex set](image)

2. A function $f$ of a single variable is quasiconcave if and only if either it is nondecreasing, or it is nonincreasing, or there is some $x^*$ such that $f$ is nondecreasing for $x < x^*$ and nonincreasing for $x > x^*$. Consider each case in turn.

- $f$ is nondecreasing
  The derivative of the function is $3ax^2 + 2bx + c$. In order for this to be nonnegative for all $x$ we certainly need $c \geq 0$ (take $x = 0$). Now, we can consider three cases separately.

  a. If $a > 0$ then the derivative is a convex quadratic, with a minimum at $x = -b/3a$. (Take the derivative of the derivative, and set it equal to zero.) The minimal value of the derivative is thus $3a\cdot(-b/3a)^2 - 2b/3a + c = -b^2/3a + c$. For the derivative to be nonnegative for all $x$ it is necessary and sufficient that this minimum be $\geq 0$, which is equivalent to $c \geq b^2/3a$.

  b. If $a = 0$ then the derivative is nonnegative for all $x$ if and only if $b = 0$ and $c \geq 0$. 


![Diagram of a non-convex set](image)
c. If \( a < 0 \) then the derivative is a concave quadratic, and hence is negative for some values of \( x \), no matter what the values of \( b \) and \( c \) are.

\( f \) is nonincreasing

The derivative of the function is \( 3ax^2 + 2bx + c \). In order for this to be nonpositive for all \( x \) we certainly need \( c \leq 0 \) (take \( x = 0 \)). Now, as in the previous case, we can consider three cases separately.

d. If \( a > 0 \) then the derivative is a convex quadratic, and hence is not nonpositive for all values of \( x \) no matter what the values of \( b \) and \( c \) are.

e. If \( a = 0 \) then the derivative is nonpositive for all \( x \) if and only if \( b = 0 \) and \( c \leq 0 \).

f. If \( a < 0 \) then the derivative is a concave quadratic, with a maximum at \( x = -b/3a \). (Take the derivative of the derivative, and set it equal to zero.) The maximal value of the derivative is thus \( 3a\cdot(b/3a)^2 - 2b/3a + c = -b/3a + c \). For the derivative to be nonpositive for all \( x \) it is necessary and sufficient that this maximum be \( \leq 0 \), which is equivalent to \( c \leq b/3a \).

\( f \) increases then decreases

If \( a > 0 \) then the derivative of \( f \) is a quadratic; since no quadratic has the property that it is positive up to some point and negative thereafter, the only possibility is that \( a = 0 \). In this case \( f \) itself is a quadratic; it increases and then decreases if and only if \( b < 0 \).

In summary, \( f \) is quasiconcave if and only if either \( a > 0 \) and \( c \geq b/3a \), or \( a < 0 \) and \( c \leq b/3a \), or \( a = 0 \) and \( b \leq 0 \).

- The bordered Hessian of \( f \) is

\[
\begin{pmatrix}
0 & -ye^x & e^{-x} \\
-ye^x & ye^x & -e^{-x} \\
e^x & -e^{-x} & 0
\end{pmatrix}
\]

- We have \( D_1(x, y) = -ye^{-2x} \leq 0 \) and \( D_2(x, y) = ye^{2x} + e^{-x}(ye^{-2x} - ye^{-2x}) = ye^{-3x} \geq 0 \). Both determinants are zero if \( y = 0 \), so while the bordered Hessian is not inconsistent with the function's being quasiconcave, it does not establish that it is in fact quasiconcave either. However, the test does show that the function is quasiconcave on the domain in which \( y \) is restricted to be positive (rather than only nonnegative).
- We can give an example involving functions of a single variable. In the figure, the top and middle functions are quasiconcave (each of them is first nondecreasing, then nonincreasing), whereas the bottom function, which is the sum of the top and middle functions, is not quasiconcave (it is not nondecreasing, is not nonincreasing, and is not nondecreasing then nonincreasing.
The function $h$ is not necessarily quasiconcave. Suppose, for example, that $f$ and $g$ are functions of a single variable, with $f(x) = -x^2$ and

$$g(x) = \begin{cases} 0 & \text{if } x < 1 \\ x & \text{if } x > 1 \end{cases}$$

(See the figure below.) Then the function $h$ increases up to $x = 0$, decreases from $x = 0$ to $x = 1$, and then increases up to $x = 2$, and is thus not quasiconcave.

No. Let $f(x) = x$ if $x \leq 1$ and $f(x) = 1$ if $x > 1$, and let $g(x) = x$ if $x \leq 1$ and $g(x) = 2 - x$ if $x > 1$. Then $f$ and $g$ are both concave, but the product, $h(x) = x^2$ if $x \leq 1$ and $h(x) = 2 - x$ if $x > 1$ is not quasiconcave (it decreases, increases, and then decreases).

a. The Hessian of the function is

$$\begin{pmatrix} 2y^2 & 4xy \\ 4xy & 2x^2 \end{pmatrix}$$
b. The determinant is negative if \((x, y) \neq (0, 0)\), so the function is not concave or convex. To test for quasiconcavity, look at the bordered Hessian:

\[
\begin{pmatrix}
0 & 2xy & 2x^2y \\
2xy & 2y^2 & 4xy \\
2x^2y & 4xy & 2x^2
\end{pmatrix}
\]

c. We have \(D_1(x, y) = -4x^2y^4\) (the determinant of the \(2 \times 2\) submatrix in the top left). This is zero if \((x, y) = (0, 0)\), so we cannot tell from this test if the function is quasiconcave. (In fact, it is.)

d. The Hessian of the function is

\[
\begin{pmatrix}
-e^x -e^x & -e^x \\
-e^x & -e^x
\end{pmatrix}
\]

e. Since \(-e^x -e^x < 0\) for all \((x, y)\) and the determinant is

\((-e^x -e^x)(-e^x) - (-e^x)(-e^x) = e^{2x} > 0\) for all \((x, y)\) the function is strictly concave, hence strictly quasiconcave. (Or you can argue that since \(e^x\) is increasing and convex and \(x + y\) is convex, \(e^{x+y}\) is convex and thus \(-e^{x+y}\) is concave, and similarly for \(-e^x\); then you need to make a separate argument for strictly concavity.)

0. We have \(f'(x_1, x_2) = (x_2)^2\), \(f'(x_1, x_2) = 2x_1x_2\), \(f''(x_1, x_2) = 0\), \(f_{x_2x_2}''(x_1, x_2) = 2x_1\), and \(f_{x_1x_2}''(x_1, x_2) = 2x_1\). Thus \(D_1(x_1, x_2) = -(x_2)^2\) and

\[
D_1(x_1, x_2) =
\begin{pmatrix}
0 & (x_2)^2 & 2x_1x_2 \\
(x_2)^2 & 0 & 2x_2 \\
2x_1x_2 & 2x_2 & 2x_1
\end{pmatrix}
\]

0. so that \(D_1(x_1, x_2) = 6x_1(x_2)^2\). Hence for all \((x_1, x_2)\) with \(x_1 > 0\) and \(x_2 > 0\) we have \(D_1(x_1, x_2) < 0\) and \(D_2(x_1, x_2) > 0\), so that \(f\) is quasiconcave.

0.

a. The curves are consistent with the function's being quasiconcave: the sets of points \((x, y)\) such that \(f(x, y) > a\), where 1, 2, and 4, are the sets of points above each curve, and each set is convex. [Note that this fact does not imply that the whole function is necessarily quasiconcave (perhaps some other upper level set is not convex); the question asks you only if the information in the figure is consistent with the function's being quasiconcave.]

b. The curves are not consistent with the function's being concave: notice that \((2, 2)\) is the midpoint of the line segment from \((1, 1)\) to \((3, 3)\), but the
value of the function at $(2, 2)$ is less than the average of the values of the function at $(1, 1)$ and at $(3, 3)$. Stated algebraically, $f(2, 2) = 2$, while 

\[(1/2) f(1, 1) + (1/2) f(3, 3) = (1/2)\cdot 1 + (1/2)\cdot 4 = (5/2),\]

so that 

\[(1/2) f(1, 1) + (1/2) f(3, 3) > f ((1/2)(1, 1) + (1/2)(3, 3)) = f(2, 2).\]

-1. Take $x$ and $y$ such that $f(x) \geq f(y)$. By concavity we have 

\[f((1-\lambda)x + \lambda y) \geq (1-\lambda) f(x) + \lambda f(y)\]

for all $\lambda$ with $0 \leq \lambda \leq 1$, so that 

\[f((1-\lambda)x + \lambda y) \geq (1-\lambda) f(y) + \lambda f(y) = f(y)\]

for all $\lambda$ with $0 \leq \lambda \leq 1$. Thus $f$ is quasiconcave.

-1. Suppose that $f(x) \geq f(y)$. We need to show that 

\[f((1-\lambda)x + \lambda y) \geq f(y)\]

for all $\lambda$ with $0 \leq \lambda \leq 1$. The fact that 

\[f((1-\lambda)x + \lambda y) \geq (1-\lambda) f(y) + \lambda f(y) = f(y)\]

for all $\lambda$ with $0 \leq \lambda \leq 1$. Thus $f$ is quasiconcave.

### 4.1 Optimization: introduction

Decision-makers (e.g. consumers, firms, governments) in standard economic theory are assumed to be "rational". That is, each decision-maker is assumed to have a preference ordering over the outcomes to which her actions lead and to choose an action, among those feasible, that is most preferred according to this ordering. We usually make assumptions that guarantee that a decision-maker's preference ordering is represented by a payoff function (sometimes called utility function), so that we can present the decision-maker's problem as one of choosing an action, among those feasible, that maximizes the value of this function. That is, we write the decision-maker's problem in the form 

\[
\max_a u(a) \text{ subject to } a \in S,
\]

where $u$ is the decision-maker's payoff function over her actions and $S$ is the set of her feasible actions.

If the decision-maker is a classical consumer, for example, then $a$ is a consumption bundle, $u$ is the consumer's utility function, and $S$ is the set of bundles of goods the consumer can afford. If the decision-maker is a classical firm then $a$ is an input-output vector, $u(a)$ is the profit the action $a$ generates, and $S$ is the set of all feasible input-output vectors (as determined by the firm's technology).

Even outside the classical theory, the actions chosen by decision-makers are often modeled as solutions of maximization problems. A firm, for example, may be assumed to maximize its sales, rather than its profit; a consumer may care not only about the bundle of goods she consumes, but also about the bundles of goods the other members of her family consumes, maximizing a function that includes these bundles as well as her own; a government may choose policies to maximize its chance of reelection.

In economic theory we sometimes need to solve a minimization problem of the form 

\[
\min_a u(a) \text{ subject to } a \in S.
\]
We assume, for example, that firms choose input bundles to minimize the cost of producing any given output; an analysis of the problem of minimizing the cost of achieving a certain payoff greatly facilitates the study of a payoff-maximizing consumer.

The next three parts of the tutorial develop tools for solving maximization and minimization problems, which are collectively known as optimization problems.

• This part discusses some basic definitions and a fundamental result regarding the existence of an optimum.
• The next part, on interior optima, focuses on conditions for solutions that are strictly inside the constraint set \( S \)---"interior" solutions.
• The third part, on equality constraints, discusses the key technique developed by Lagrange for finding the solutions of problems in which \( S \) is the set of points that satisfy a set of equations.
• The last part, on the Kuhn-Tucker conditions for problems with inequality constraints, discusses a set of conditions that may be used to find the solutions of any problem in which \( S \) is the set of points that satisfy a set of inequalities. These conditions encompass both the conditions for interior optima and those developed by Lagrange. The last section is a summary of the conditions in the previous parts.

### 4.2 Optimization: definitions

The optimization problems we study take the form

\[
\text{max. } f(x) \text{ subject to } x \in S
\]

where \( f \) is a function, \( x \) is a vector of variables \((x_1, \ldots, x_n)\), and \( S \) is a set of \( n \)-vectors. We call \( f \) the objective function, \( x \) the choice variable, and \( S \) the constraint set or opportunity set.

To be very clear about what we mean by a maximum, here is a precise definition.

**Definition**

The value \( x^* \) of the variable \( x \) solves the problem

\[
\text{max. } f(x) \text{ subject to } x \in S
\]

if

\[
f(x) \leq f(x^*) \text{ for all } x \in S.
\]

In this case we say that \( x^* \) is a maximizer of the function \( f \) subject to the constraint \( x \in S \), and that \( f(x^*) \) is the maximum (or maximum value) of the function \( f \) subject to the constraint \( x \in S \).

As an example, both \( x^* \) and \( x^{**} \) are maximizers of \( f \) subject to the constraint \( x \in S \) for the function of one variable in the following figure.
What about the point $x'$? This point is decidedly not a maximizer, because $f(x^*) > f(x')$, for example. But it is a maximum among the points close to it. We call such a point a local maximizer. In the following definition, the distance between two points $x$ and $x'$ is the square root of $\sum_{i=1}^n (x_i - x'_i)^2$ (sometimes called the Euclidean distance between $x$ and $x'$).

**Definition**

The variable $x^*$ is a **local maximizer** of the function $f$ subject to the constraint $x \in S$ if there is a number $\varepsilon > 0$ such that $f(x) \leq f(x^*)$ for all $x \in S$ for which the distance between $x$ and $x^*$ is at most $\varepsilon$.

For the function $f$ in the figure above we have $f(x) \leq f(x')$ for all $x$ between $x_1$ and $x_2$, where $x_1 - x' = x_2 - x'$, so the point $x'$ satisfies this definition, and is thus a local maximizer of $f$.

Sometimes we refer to a maximizer as a **global maximizer** to emphasize that it is not only a local maximizer. Every global maximizer is, in particular, a local maximizer.

In economic theory we are almost always interested in global maximizers, not merely local maximizers. For example, the standard theory is that a consumer chooses the bundle she most prefers among all those that are available; she is not satisfied by a bundle that is merely better than the other bundles that are nearby. Similarly, a firm is assumed to choose the input-output vector that maximizes its profit among all those that are feasible; it is not satisfied by an input-output vector that merely yields a higher profit than does similar vectors.

**Transforming the objective function**

Let $g$ be a strictly increasing function of a single variable. (That is, if $z' > z$ then $g(z') > g(z)$.) Then the set of solutions to the problem

$max, \ f(x) \text{ subject to } x \in S$

is identical to the set of solutions to the problem

$max, \ g(f(x)) \text{ subject to } x \in S$.

Why? If $x^*$ is a solution to the first problem then by definition $f(x) \leq f(x^*)$ for all $x \in S$.

But if $f(x) \leq f(x^*)$ then $g(f(x)) \leq g(f(x^*))$, so that $g(f(x)) \leq g(f(x^*))$ for all $x \in S$.

Hence $x^*$ is a solution of the second problem.
This fact is sometimes useful: for example, if the objective function is \( f(x_1, x_2) = x_1^\alpha x_2^\beta \), it may be easier to work with the logarithm of this function, namely \( \alpha \ln x_1 + \beta \ln x_2 \). Since \( \log z \) is an increasing function, the set of solutions of the problem in which \( x_1^\alpha x_2^\beta \) is the objective function is the same as the set of solutions of the problem in which \( \alpha \ln x_1 + \beta \ln x_2 \) is the objective function.

**Minimization problems**

What about minimization problems? Any discussion of maximization problems encompasses minimization problems because of a simple fact: any minimization problem can be turned into a maximization problem by taking the negative of the objective function. That is, the problem

\[
\min \ f(x) \text{ subject to } x \in S
\]

is equivalent---in particular, has the same set of solutions---as the problem

\[
\max -f(x) \text{ subject to } x \in S.
\]

Thus we can solve any minimization problem by taking the negative of the objective function and apply the results for maximization problems.

In this tutorial I emphasize maximization problems, but usually state results separately both for maximization problems and for minimization problems.

### 4.2 Exercises on basic definitions for optimization theory

1. Consider the function \( f \) of a single variable defined by \( f(x) = -x - 1 \) for \( x < -1 \), \( f(x) = 0 \) for \(-1 \leq x \leq 1\), and \( f(x) = x - 1 \) for \( x > 1 \). Is the point \( x = 0 \) a global maximizer and/or a global minimizer and/or a local maximizer and/or a local minimizer of \( f \)?

2. Consider the function \( f \) of a single variable defined by \( f(x) = x + 1 \) for \( x < -1 \), \( f(x) = 0 \) for \(-1 \leq x \leq 1\), and \( f(x) = x - 1 \) for \( x > 1 \). Is the point \( x = 0 \) a global maximizer and/or a global minimizer and/or a local maximizer and/or a local minimizer of \( f \)?

### 4.2 Solutions to exercises on basic definitions for optimization theory

1. The point \( x = 0 \) is not a global maximizer (\( f(2) = 1 > f(0) = 0 \), for example), but is a global minimizer (\( f(x) \geq f(0) = 0 \) for all \( x \)), and is both a local maximizer (\( f(x) \leq f(0) = 0 \) for all \( x \) with \(-1 \leq x \leq 1\)) and a local minimizer.

2. The point \( x = 0 \) is not a global maximizer (\( f(2) = 1 > f(0) = 0 \), for example), or a global minimizer (\( f(-2) = -1 < f(0) = 0 \), for example). It is both a local maximizer (\( f(x) \leq f(0) = 0 \) for all \( x \) with \(-1 \leq x \leq 1\)) and a local minimizer.
4.3 Existence of an optimum

Let \( f \) be a function of \( n \) variables defined on the set \( S \). The problems we consider take the form
\[
\max f(x) \text{ subject to } x \in S
\]
where \( x = (x_1, \ldots, x_n) \).

Before we start to think about how to find the solution to a problem, we need to think about whether the problem has a solution. Here are some specifications of \( f \) and \( S \) for which the problem does not have any solution.

- \( f(x) = x, \quad S = [0, \infty) \) (i.e. \( S \) is the set of all nonnegative real numbers). In this case, \( f \) increases without bound, and never attains a maximum.
- \( f(x) = 1 - 1/x, \quad S = [1, \infty) \). In this case, \( f \) converges to the value 1, but never attains this value.
- \( f(x) = x, \quad S = (0, 1) \). In this case, the points 0 and 1 are excluded from \( S \) (which is an open interval). As \( x \) approaches 1, the value of the function approaches 1, but this value is never attained for values of \( x \) in \( S \), because \( S \) excludes \( x = 1 \).
- \( f(x) = x \) if \( x < 1/2 \) and \( f(x) = x - 1 \) if \( x \geq 1/2; \quad S = [0, 1] \). In this case, as \( x \) approaches 1/2 the value of the function approaches 1/2, but this value is never attained, because at \( x = 1/2 \) the function jumps down to \(-1/2\).

The difficulties in the first two cases are that the set \( S \) is unbounded; the difficulty in the third case is that the interval \( S \) is open (does not contain its endpoints); and the difficulty in the last case is that the function \( f \) is discontinuous. If \( S \) is a closed interval \([a, b]\) (where \( a \) and \( b \) are finite) and \( f \) is continuous, then none of the difficulties arise.

For functions of many variables, we need to define the concept of a bounded set.

**Definition**

The set \( S \) is bounded if there exists a number \( k \) such that the distance of every point in \( S \) from the origin is at most \( k \).

A bounded set does not extend "infinitely" in any direction.

**Example**

The set \([-1, 100]\) is bounded, because the distance of any point in the set from 0 is at most 100. The set \([0, \infty)\) is not bounded, because for any number \( k \), the number \( 2k \) is in the set, and the distance of \( 2k \) to 0 is \( 2k \) which exceeds \( k \).
Example
The set \( \{(x, y): x^2 + y^2 \leq 4\} \) is bounded, because the distance of any point in the set from \((0, 0)\) is at most 2.

Example
The set \( \{(x, y): xy \leq 1\} \) is not bounded, because for any number \(k\) the point \((2k, 0)\) is in the set, and the distance of this point from \((0, 0)\) is \(2k\), which exceeds \(k\).

We say that a set that is closed and bounded is **compact**.

The following result generalizes the observations in the examples at the top of this page.

**Proposition (Extreme value theorem)**
A continuous function on a compact set attains both a maximum and a minimum on the set.

Note that the requirement of boundedness is on the set, not the function. Requiring that the function be bounded is not enough, as the **second example** at the top of this page shows.

Note also that the result gives only a **sufficient** condition for a function to have a maximum. **If** a function is continuous and is defined on a compact set **then** it definitely has a maximum and a minimum. The result does **not** rule out the possibility that a function has a maximum and/or minimum if it is not continuous or is not defined on a compact set. (Refer to the **section on logic** if you are unclear on this point.)

### 4.3 Exercises on the existence of an optimum

1. For each of the following functions, determine (i) whether the extreme value theorem implies that the function has a maximum and a minimum and (ii) if the extreme value theorem does not apply, whether the function does in fact have a maximum and/or a minimum.
   
   a. \( x^2 \) on the interval \([-1,1]\)
   
   b. \( x^2 \) on the interval \((-1,1)\)
   
   c. \( \mid x \mid \) on the interval \([-1,\infty)\)
   
   d. \( f(x) \) defined by \( f(x) = 1 \) if \( x < 0 \) and \( f(x) = x \) if \( x \geq 0 \), on the interval \([-1,1]\).
   
   e. \( f(x) \) defined by \( f(x) = 1 \) if \( x < 0 \) and \( f(x) = x \) if \( x \geq 0 \), on the interval \((-\infty,\infty)\).
   
   f. \( f(x) \) defined by \( f(x) = x^2 \) if \( x < 0 \) and \( f(x) = x \) if \( x \geq 0 \) on the interval \([-1,1]\).

2. Does the extreme value theorem imply that the problem

\[
\max u(x) \text{ subject to } p \cdot x \leq w \text{ and } x \geq 0,
\]

implies...
where $x$ is an $n$-vector, $u$ is a continuous function, and $p > 0$ (an $n$-vector) and $w > 0$ (a scalar) are parameters, has a solution? If not, specify a function $u$ for which the problem has a solution, if there is such a function; and specify a function $u$ for which the problem does not have a solution, if there is such a function. (The problem may be interpreted as the optimization problem of a consumer with a utility function $u$ and income $w$, facing the price vector $p$.)

3. Does the extreme value theorem imply that the problem

$$ \max_p f(x) - w \cdot x \text{ subject to } x \geq 0, $$

where $x$ is an $n$-vector, $f$ is a continuous function, and $p > 0$ (a scalar) and $w > 0$ (an $n$-vector) are parameters, has a solution? If not, specify a function $f$ and values of $p$ and $w$ for which the problem has a solution, if such exist; and specify a function $f$ and values for $p$ and $w$ for which the problem does not have a solution, if such exist. (The problem may be interpreted as the optimization problem of a firm with production function $f$ facing the input price vector $w$ and the price of output $p$.)

4. For each of the following problems, determine whether the extreme value theorem implies that a solution exists. (You need to give reasons for your answers! Note that for cases in which the extreme value theorem does not apply, you are not asked to determine whether there is in fact a solution.)

   a. $\max x^2 + y^2 \text{ subject to } x^2 + 2y^2 \leq 1.$
   b. $\max f(x) \text{ subject to } x \geq 0,$ where $f(x) = -x^2$ if $x \leq -1$ and $f(x) = -1$ if $x > -1.$
   c. $\max f(x) \text{ subject to } 0 \leq x \leq 1,$ where $f(x) = x^2$ if $0 \leq x \leq 1/2$ and $f(x) = 1/2$ if $1/2 < x \leq 1.$

### 4.3 Solutions to exercises on the existence of an optimum

1. a. EVT applies.
   b. EVT does not apply, because interval is not closed. Function has minimum of 0, but no maximum.
   c. EVT does not apply, because interval is not bounded. Function has minimum of 0, but no maximum.
   d. EVT does not apply, because function is not continuous. Function has minimum of 0 (attained at 0) and maximum of 1 (attained at all points in $[-1,0]$ and at 1).
   e. EVT does not apply, because function is not continuous and interval is not bounded. Function has minimum of 0, but not maximum.
   f. EVT applies.

2. The objective function is continuous. The constraint set is the set of all nonnegative vectors on or below a hyperplane (the multidimensional analogue of
a line in two dimensions and a plane in three dimensions). (Draw a diagram in the two dimensional case.) Thus it is compact. Hence the extreme value theorem implies that the problem has a solution.

3. The extreme value theorem does not imply that the problem has a solution, because the constraint set is not bounded.

If \( p = 1 \), \( f(x) = x^{1/2} \), and \( w = 1 \), then the problem has a solution. (Draw a graph of the objective function.)

If \( p = 1 \), \( f(x) = x^2 \), and \( w = 1 \), then the problem does not have a solution. (Draw a graph of the objective function.)

4.

a. The objective function is continuous, and the constraint set is closed and bounded, therefore the extreme value theorem implies the problem has a solution.

b. The constraint set is not bounded, so the extreme value theorem does not apply.

c. The objective function is not continuous, so the extreme value theorem does not apply.
5.1 Necessary conditions for an interior optimum

One variable

From your previous study of mathematics, you probably know that if the function $f$ of a single variable is differentiable then there is a relationship between the solutions of the problem

$$\max_{x \in I} f(x),$$

where $I$ is an interval of numbers, and the points at which the first derivative of $f$ is zero. What precisely is this relationship?

We call a point $x$ such that $f'(x) = 0$ a stationary point of $f$. Consider the cases in the three figures.

- In the left figure, the unique stationary point $x^*$ is the global maximizer.
- In the middle figure, there are three stationary points: $x^*, x', x''$. The point $x^*$ is the global maximizer, while $x'$ is a local (though not global) minimizer and $x''$ is a local (but not global) maximizer.
- In the right figure, there are two stationary points: $x'$ and $x''$. The point $x'$ in neither a local maximizer nor a local minimizer; $x''$ is a global minimizer.

We see that

- a stationary point is not necessarily a global maximizer, or even a local maximizer, or even a local optimizer of any sort (maximizer or minimizer) (consider $x'$ in the right-hand figure)
- a global maximizer is not necessarily a stationary point (consider $a$ in the right-hand figure).

That is, being a stationary point is **neither** a necessary condition nor a sufficient condition for solving the problem. So what is the relation between stationary points and maximizers?
Although a maximizer may not be a stationary point, the only case in which it is not is when it is one of the endpoints of the interval \( I \) on which \( f \) is defined. That is, any point \textit{interior} to this interval that is a maximum must be a stationary point.

**Proposition**

Let \( f \) be a differentiable function of a single variable defined on the interval \( I \). If a point \( x \) in the interior of \( I \) is a local or global maximizer or minimizer of \( f \) then \( f'(x) = 0 \).

This result gives a \textit{necessary} condition for \( x \) to be a maximizer (or a minimizer) of \( f \): \textit{if} it is a maximizer (or a minimizer) and is between \( a \) and \( b \) then \( x \) is a stationary point of \( f \). The condition is obviously \textit{not} \textit{sufficient} for a point to be a maximizer—the condition is satisfied also, for example, at points that are minimizers. The first-derivative is involved, so we refer to the condition as a \textit{first-order condition}.

Thus among all the points in the interval \( I \), only the endpoints (if any) and the stationary points of \( f \) can be maximizers of \( f \). Most functions have a relatively small number of stationary points, so the following procedure to find the maximizers is useful.

**Procedure for solving a single-variable maximization problem on an interval**

Let \( f \) be a differentiable function of a single variable and let \( I \) be an interval of numbers. If the problem \( \max_x f(x) \) subject to \( x \in I \) has a solution, it may be found as follows.

- Find all the stationary points of \( f \) (the points \( x \) for which \( f'(x) = 0 \) that are in the constraint set \( S \), and calculate the values of \( f \) at each such point.
- Find the values of \( f \) at the endpoints, if any, of \( I \).
- The points \( x \) you have found at which the value \( f(x) \) is largest are the maximizers of \( f \).

The variant of this procedure in which the last step involves choosing the points \( x \) at which \( f(x) \) is smallest may be used to solve the analogous minimization problem.

**Example**

Consider the problem \( \max_x x^2 \) subject to \( x \in [-1, 2] \). This problem satisfies the conditions of the \textit{extreme value theorem}, and hence has a solution. Let \( f(x) = x^2 \). We have \( f'(x) = 2x \), so the function has a single stationary point, \( x = 0 \), which is in the constraint set. The value of the function at this point is \( f(0) = 0 \). The values of \( f \) at the endpoints of the interval on which it is defined are \( f(-1) = 1 \) and \( f(2) = 4 \). Thus the global maximizer of the function on \([-1, 2]\) is \( x = 2 \) and the global minimizer is \( x = 0 \).
Example
Consider the problem
\[ \max x - x^2 \text{ subject to } x \in (-\infty, \infty). \]
This problem does not satisfy the conditions of the extreme value theorem, so that the theorem does not tell us whether it has a solution. Let \( f(x) = -x^2 \). We have \( f'(x) = -2x \), so that the function has a single stationary point, \( x = 0 \), which is in the constraint set. The constraint set has no endpoints, so \( x = 0 \) is the only candidate for a solution to the problem. We conclude that if the problem has a solution then the solution is \( x = 0 \). In fact, the problem does have a solution: we have \( f(x) \leq 0 \) for all \( x \) and \( f(0) = 0 \), so the solution is indeed \( x = 0 \).

Example
Consider the problem
\[ \max x^2 \text{ subject to } x \in (-\infty, \infty). \]
Like the problem in the previous example, this problem does not satisfy the conditions of the extreme value theorem, so that the theorem does not tell us whether it has a solution. Let \( f(x) = x^2 \). We have \( f'(x) = 2x \), so that the function has a single stationary point, \( x = 0 \), which is in the constraint set. The constraint set has no endpoints, so \( x = 0 \) is the only candidate for a solution to the problem. We conclude that if the problem has a solution then the solution is \( x = 0 \). In fact, the problem does not have a solution: the function \( f \) increases without bound as \( x \) increases (or decreases) without bound.

Many variables
Consider a maximum of a function of two variables. At this maximum the function must decrease in every direction (otherwise the point would not be a maximum!). In particular, the maximum must be a maximum along a line parallel to the \( x \)-axis and also a maximum along a line parallel to the \( y \)-axis. Hence, given the result for a function of a single variable, at the maximum both the partial derivative with respect to \( x \) and the partial derivative with respect to \( y \) must be zero. Extending this idea to many dimensions gives us the following result, where \( f', \) is the partial derivative of \( f \) with respect to its \( i \)th argument.

**Proposition**
Let \( f \) be a differentiable function of \( n \) variables defined on the set \( S \). If the point \( x \) in the interior of \( S \) is a local or global maximizer or minimizer of \( f \) then
\[ f'(x) = 0 \text{ for } i = 1, \ldots, n. \]

As for the analogous result for functions of a single variable, this result gives a necessary condition for a maximum (or minimum): if a point is a maximizer then it satisfies the condition. As before, the condition is called a first-order condition. Any point at which all the partial derivatives of \( f \) are zero is called a stationary point of \( f \).

As for functions of a single variable, the result tells us that the only points that can be global maximizers are either stationary points or boundary points of the set \( S \). Thus the following procedure locates all global maximizers and global minimizers of a differentiable function.
Procedure for solving a many-variable maximization problem on a set

Let \( f \) be a differentiable function of \( n \) variables and let \( S \) be a set of \( n \)-vectors. If the problem \( \max_x f(x) \) subject to \( x \in S \) has a solution, it may be found as follows.

- Find all the stationary points of \( f \) (the points \( x \) for which \( f'(x) = 0 \) for \( i = 1, \ldots, n \)) in the constraint set \( S \) and calculate the value of \( f \) at each point.
- Find the largest and smallest values of \( f \) on the boundary of \( S \).
- The points \( x \) you have found at which the value of \( f \) is largest are the maximizers of \( f \).

This method is much less generally useful than the analogous method for functions of a single variable because for many problems finding the largest and smallest values of \( f \) on the boundary of \( S \) is difficult. For this reason, we devote considerable attention to other, better methods for finding maxima and minima of maximization problems with constraints, in the next two parts.

Here are some examples, however, where the method may be fairly easily applied.

Example

Consider the problem

\[
\max_{x, y} \left[ -(x - 1)^2 - (y + 2)^2 \right] \text{ subject to } -\infty < x < \infty \text{ and } -\infty < y < \infty.
\]

This problem does not satisfy the conditions of the extreme value theorem (because the constraint set is not bounded), so the theorem does not tell us whether the problem has a solution. The first-order conditions are

\[
\begin{align*}
-2(x - 1) &= 0 \\
-2(y + 2) &= 0,
\end{align*}
\]

which have a unique solution, \((x, y) = (1, -2)\). The constraint set has no boundary points, so we conclude that if the problem has a solution, this solution is \((x, y) = (1, -2)\). In fact, the problem does have a solution, because the value of the objective function at \((1, -2)\) is 0, and its value at any point is nonpositive.

Example

Consider the problem

\[
\max_{x, y} \left[ (x - 1)^2 + (y - 1)^2 \right] \text{ subject to } 0 \leq x \leq 2 \text{ and } -1 \leq y \leq 3.
\]

This problem satisfies the conditions of the extreme value theorem, and hence has a solution. The first-order conditions are

\[
\begin{align*}
-2(x - 1) &= 0 \\
-2(y - 2) &= 0,
\end{align*}
\]

which have a unique solution, \((x, y) = (1, 1)\), which is in the constraint set. The value of the objective function at this point is 0.

Now consider the behavior of the objective function on the boundary of the constraint set, which is a rectangle.
• If \( x = 0 \) and \(-1 \leq y \leq 3\) then the value of the objective function is \(1 + (y - 1)^2\). The problem of finding \( y \) to maximize this function subject to \(-1 \leq y \leq 3\) satisfies the conditions of the extreme value theorem, and thus has a solution. The first-order condition is \(2(y - 1) = 0\), which has a unique solution \( y = 1 \), which is in the constraint set. The value of the objective function at this point is 1. On the boundary of the set \{(0, y); -1 \leq y \leq 3\}—namely at the points (0, -1) and (0, 3)—the value of the objective function is 5. Thus on this part of the boundary, the points (0, -1) and (0, 3) are the only candidates for a solution of the original problem.

• A similar analysis leads to the conclusion that the points (2, -1) and (2, 3) are the only candidates for a maximizer on the part of the boundary for which \( x = 2 \) and \(-1 \leq y \leq 3\), the points (0, -1) and (2, -1) are the only candidates for a maximizer on the part of the boundary for which \(0 \leq x \leq 2\) and \(y = -1\), and the points (0, 3) and (2, 3) are the only candidates for a maximizer on the part of the boundary for which \(0 \leq x \leq 2\) and \(y = 3\).

• The value of the objective function at all these candidates for a solution on the boundary of the constraint set is 5.

Finally, comparing the values of the objective function at the candidates for a solution that are (a) interior to the constraint set (namely (1, 1)) and (b) on the boundary of the constraint set, we conclude that the problem has four solutions, (0, -1), (0, 3), (2, -1), and (2, 3).

**Example**

Consider the problems

max\(, x^2 + y^2 + y - 1 \) subject to \(x^2 + y^2 \leq 1\)

and

min\(, x^2 + y^2 + y - 1 \) subject to \(x^2 + y^2 \leq 1\).

In each case the constraint set, \{(x, y): x^2 + y^2 \leq 1\}, is compact. The objective function is continuous, so by the extreme value theorem, the problem has a solution.

We apply the procedure as follows, denoting the objective function by \( f \).

• We have \( f_1(x, y) = 2x \) and \( f_2(x, y) = 2y + 1 \), so the stationary points are the solutions of \(2x = 0\) and \(2y + 1 = 0\). Thus the function has a single stationary point, \((x, y) = (0, -1/2)\), which is in the constraint set. The value of the function at this point is \( f(0, -1/2) = -5/4\).

• The boundary of the constraint set is the set of points \((x, y)\) such that \(x^2 + y^2 = 1\), as shown in the following figure.
Thus for a point \((x, y)\) on the boundary we have \(f(x, y) = x^2 + 1 - x^2 + y - 1 = y\). We have \(-1 \leq y \leq 1\) on the boundary, so the maximum of the function on the boundary is 1, which is achieved at \((x, y) = (0, 1)\), and the minimum is \(-1\), achieved at \((x, y) = (0, -1)\).

- Looking at all the values we have found, we see that the global maximum of \(f\) is 1, achieved at \((0, 1)\), and the global minimum is \(-5/4\), achieved at \((0, -1/2)\).

Notice that the reasoning about the behavior of the function on the boundary of the constraint set is straightforward in this example because we are able easily to express the value of the function on the boundary in terms of a single variable \((y)\). In many other problems

### 5.1 Exercises on a necessary condition for an interior optimum

1. Find all the global maximizers and minimizers of the following functions of a single variable.
   a. \(f(x) = 1 - x\) on the interval \([0, 1]\).
   b. \(f(x) = (x - 1/3)^2\) on the interval \([0, 1]\).
   c. \(f(x) = x^3 - 2x^2 + x\) on the interval \([0, 2]\).
2. Solve the problem \(\max_{x, y} x^2 + 2y^2 - x\) subject to \(x^2 + y^2 \leq 1\) and the corresponding minimization problem using the procedure described on the main page. [When studying the behavior of the function on the boundary of the constraint set, use the definition of the boundary to express the value of the function in terms of a single variable.]
3. Solve the problem \( \text{max}_{x, y} 3 + x - x^2 - y \) subject to \( x + y^2 \leq 1 \) and \( x \geq 0 \) and the corresponding minimization problem using the procedure described on the main page.

4. A consumer has the utility function \( x_1 x_2 \), has income \( y > 0 \), and faces the prices \( p_t > 0 \) and \( p_s > 0 \). She is required to spend all her income. Formulate her problem as a constrained maximization problem in which the variables are \( x_1 \) and \( x_2 \) and the constraints are the budget constraint, \( x_1 \geq 0 \), and \( x_2 \geq 0 \). Transform this problem into a maximization problem in the single variable \( x_1 \), which is constrained to be in an interval, by isolating \( x_2 \) in the budget constraint and substituting it into the utility function. Solve this problem using the procedure described on the main page, thus finding the bundle \((x_1, x_2)\) that maximizes the consumer's utility subject to her budget constraint.

5.1 Solutions to exercises on a necessary condition for an interior optimum

1. In all three cases, the function \( f \) is continuous and the constraint set is compact, so the extreme value theorem implies that the problem has a solution.
   a. We have \( f'(x) = -1 \), so the function has no stationary points. Thus the only candidates for maximizers and minimizers are the endpoints of the interval, 0 and 1. We have \( f(0) = 1 \) and \( f(1) = 0 \), so the global maximizer is \( x = 0 \), yielding a value for the function of 1, and the global minimizer is \( x = 1 \), yielding the value 0.
   b. We have \( f'(x) = 2(x - 1/3) \), so that the function has a single stationary point, \( x = 1/3 \), at which the value of the function is 0. We have also \( f(0) = 1/9 \) and \( f(1) = 4/9 \), so the global maximizer is \( x = 1 \) and the global minimizer is \( x = 1/3 \); the global maximum is \( f(1) = 4/9 \) and the global minimum is \( f(1/3) = 0 \).
   c. We have \( f'(x) = 3x^2 - 4x + 1 \), so \( f'(x) = 0 \) if and only if \((3x - 1)(x - 1) = 0\), or if and only if \( x = 1/3 \) or \( x = 1 \). The values of \( f \) at these two points are \( f(1/3) = 4/27 \) and \( f(1) = 0 \). The values of \( f \) at the endpoints of its domain are \( f(0) = 0 \) and \( f(2) = 2 \). Thus the global maximizer is 2 (with a value of 2) and the global minimizers are 0 and 1 (with a value of 0).

2. Let \( f(x, y) = x^2 + 2y^2 - x \).

**Stationary points of** \( f \): We have \( f'(x, y) = 2x - 1 \) and \( f'(x, y) = 4y \). Thus the function has a single stationary point, \((x, y) = (1/2, 0)\). The value of the function at this point is \((1/2)^2 - 1/2 = -1/4\).

**Largest and smallest values of** \( f \) **on the boundary of the constraint set**: Now consider the behavior of the function on the boundary of the constraint set, where \( x^2 + y^2 = 1 \). This boundary is shown in the following figure.
On this boundary we have \( y^2 = 1 - x^2 \), so that \( f(x, y) = x^2 + 2(1 - x^2) - x = 2 - x - x^2 \). The values of \( x \) compatible with \( x^2 + y^2 = 1 \) are \(-1 \leq x \leq 1\) (see the figure). To find the maximum and minimum of the function on the boundary of the constraint set we thus need to find the values of \( x \) in the interval \([-1, 1]\) that maximize and minimize \( g(x) = 2 - x - x^2 \). This function is shown in the following figure.

We have \( g'(x) = -1 - 2x \), so that \( g \) has a single stationary point, \( x = -1/2 \). We have \( g(-1) = 2 \), \( g(-1/2) = 9/4 \), and \( g(1) = 0 \). Thus the maximum of \( f \) on the boundary of the constraint set is \( 9/4 \), attained at \((x, y) = (-1/2, \sqrt{3}/2)\) and \((x, y) = (-1/2, -\sqrt{3}/2)\), and the minimum of \( f \) on this set is \( 0 \), attained at \((x, y) = (1, 0)\).

(Recall that \( y^2 = 1 - x^2 \), or \( y = (1 - x^2)^{1/2} \) on the boundary of the constraint set.)

**Solution of problems:** We conclude that the maximum of \( f \) on its domain is \( 9/4 \), attained at \((-1/2, \sqrt{3}/2)\) and \((-1/2, -\sqrt{3}/2)\), and the minimum is \(-1/4 \), attained at \((1/2, 0)\).

3. Let \( f(x, y) = 3 + x^2 - x^2 - y^2 \).
Stationary points of $f$: We have $f'(x, y) = 3x^2 - 2x = x(3x - 2)$ and $f''(x, y) = 2y$. Thus the function has two stationary points, $(x, y) = (0, 0)$ and $(x, y) = (2/3, 0)$. The values of the function at these points are 3 and $77/27$.

Largest and smallest values of $f$ on the boundary of the constraint set: Now consider the behavior of the function on the boundary of the constraint set, the set of points $(x, y)$ such that either $x^2 + y^2 = 1$ and $x \geq 0$, or $x = 0$ and $-1 \leq y \leq 1$.

a. If $x^2 + y^2 = 1$ and $x \geq 0$ we have $f(x, y) = 3 + x^2 - 1 + x^2 = 2 + x^2$. This function is maximized when $x = 1$, when its value is 3, and minimized when $x = 0$, when its value is 2.

b. If $x = 0$ and $-1 \leq y \leq 1$ then $f(x, y) = 3 - y^2$, which is maximized when $y = 0$, when its value is 3, and minimized when $y = 1$ and when $y = -1$, when its value is 2.

We conclude that the maximum of $f$ on the boundary of the constraint set is 3, which is attained at $(x, y) = (1, 0)$, and the minimum is 2, which is attained at $(x, y) = (0, 1)$ and $(x, y) = (0, -1)$.

Solutions of problems: Putting these observations together, the maximum of the function on its domain is 3, attained at $(0, 0)$ and at $(1, 0)$, and the minimum is 2, attained at $(0, -1)$ and at $(0, 1)$.

- The consumer's maximization problem is

$$\max_{x_1, x_2} x_1 x_2 \text{ subject to } p_1 x_1 + p_2 x_2 = y, \ x_1 \geq 0, \text{ and } x_2 \geq 0.$$ 

From the budget constraint we have $x_2 = (y - p_1 x_1)/p_2$, so that we can write the problem as

$$\max_{x_1} (y - p_1 x_1)/p_2 \text{ subject to } 0 \leq x_1 \leq y/p_1.$$ 

The derivative of this function with respect to $x_1$ is

$$\frac{(y - p_1 x_1) p_2 - p_1 x_1 p_2}{p_2^2}.$$ 

Thus the function has a single stationary point, $x_1 = y/2p_1$, at which its value is $(y/2p_1)(y/2p_2)$. The values of the function at $x_1 = 0$ and at $x_1 = y/p_1$ are both 0, so we conclude that the function has a single global maximizer, $x_1 = y/2p_1$.

Given that $x_1 = (y - p_1 x_1)/p_2$, we conclude that the original problem has a single solution, $(x_1, x_2) = (y/2p_1, y/2p_2)$. 

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5.2 Local optima

**One variable**

Occasionally we are interested only in the local maximizers or minimizers of a function. We may be able to tell whether a stationary point is a local maximum, a local minimum, or neither by examining the second derivative of the function at the stationary point.

**Proposition (Second-order conditions for optimum of function of one variable)**

Let $f$ be a function of a single variable with continuous first and second derivatives, defined on the interval $I$. Suppose that $x^*$ is a stationary point of $f$ in the interior of $I$ (so that $f'(x^*) = 0$).

- If $f''(x^*) < 0$ then $x^*$ is a local maximizer.
- If $x^*$ is a local maximizer then $f''(x^*) \leq 0$.
- If $f''(x^*) > 0$ then $x^*$ is a local minimizer.
- If $x^*$ is a local minimizer then $f''(x^*) \geq 0$.

If $f''(x^*) = 0$ then we don’t know, without further investigation, whether $x^*$ is a local maximizer or local minimizer of $f$, or neither (check the functions $x^4$, $-x^4$, and $x^3$ at $x = 0$). In this case, information about the signs of the higher order derivatives may tell us whether a point is a local maximum or a local minimum. In practice, however, these conditions are rarely useful, so I do not discuss them.

**Many variables**

As for a function of a single variable, a stationary point of a function of many variables may be a local maximizer, a local minimizer, or neither, and we may be able to distinguish the cases by examining the second-order derivatives of the function at the stationary point.

Let $(x_0, y_0)$ be a stationary point of the function $f$ of two variables. Suppose it is a local maximizer. Then certainly it must be a maximizer along the two lines through $(x_0, y_0)$ parallel to the axes. Using the theory for functions of a single variable, we conclude that

$$f_{xx}''(x_0, y_0) \leq 0 \quad \text{and} \quad f_{yy}''(x_0, y_0) \leq 0,$$

where $f''_{ij}$ denotes the second partial derivative of $f$ with respect to its $i$th argument, then with respect to its $j$th argument.

However, even the variant of this condition in which both inequalities are strict is not sufficient for $(x_0, y_0)$ to be a maximizer, as the following example shows.

**Example**

Consider the function $f(x, y) = 3xy - x^3 - y^3$. The first-order conditions are

$$f_x'(x, y) = 3y - 2x = 0$$
$$f_y'(x, y) = 3x - 2y = 0$$

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so that \( f \) has a single stationary point, \((x, y) = (0, 0)\). Now,

\[
\begin{align*}
    f_{11}''(0, 0) &= -2 \leq 0 \\
    f_{22}''(x, y) &= -2 \leq 0.
\end{align*}
\]

But \((0, 0)\) is not a local maximizer: at \((0, 0)\) the value of the function is 0, but at \((\varepsilon, \varepsilon)\) with \(\varepsilon > 0\) the value of the function is \(3\varepsilon^2 - \varepsilon^2 - \varepsilon^2 = \varepsilon^2\), which is positive (and hence exceeds \( f(0, 0) = 0 \)) no matter how small \(\varepsilon\) is.

This example shows that we cannot determine the nature of a stationary point of a function \( f \) of two variables by looking only at the partial derivatives \( f_{11}'' \) and \( f_{22}'' \) at the stationary point.

The next result gives a condition that involves the definiteness of the Hessian of the function, and thus all the cross-partials. The result assumes that all the second-order partial derivatives \( f_{ij}'' \) are continuous for all \( x \) in some set \( S \), so that by Young’s theorem we have \( f_{ij}''(x) = f_{ji}''(x) \) for all \( x \in S \), and hence the Hessian is symmetric. (The condition on \( f \) is satisfied, for example, by any polynomial.)

**Proposition (Second-order conditions for optimum of function of many variables)**

Let \( f \) be a function of \( n \) variables with continuous partial derivatives of first and second order, defined on the set \( S \). Suppose that \( x^* \) is a stationary point of \( f \) in the interior of \( S \) (so that \( f_i'(x^*) = 0 \) for all \( i \)).

- If \( H(x^*) \) is negative definite then \( x^* \) is a local maximizer.
- If \( x^* \) is a local maximizer then \( H(x^*) \) is negative semidefinite.
- If \( H(x^*) \) is positive definite then \( x^* \) is a local minimizer.
- If \( x^* \) is a local minimizer then \( H(x^*) \) is positive semidefinite.

An implication of this result is that if \( x^* \) is a stationary point of \( f \) then

- if \( H(x^*) \) is negative definite then \( x^* \) is a local maximizer
- if \( H(x^*) \) is negative semidefinite, but neither negative definite nor positive semidefinite, then \( x^* \) is not a local minimizer, but might be a local maximizer
- if \( H(x^*) \) is positive definite then \( x^* \) is a local minimizer
- if \( H(x^*) \) is positive semidefinite, but neither positive definite nor negative semidefinite, then \( x^* \) is not a local maximizer, but might be a local minimizer
- if \( H(x^*) \) is neither positive semidefinite nor negative semidefinite then \( x^* \) is neither a local maximizer nor a local minimizer.

For a function \( f \) of two variables, the Hessian is

\[
\begin{pmatrix}
    f_{11}''(x^*) & f_{12}''(x^*) \\
    f_{21}''(x^*) & f_{22}''(x^*)
\end{pmatrix}.
\]

This matrix is negative definite if \( f_{11}''(x^*) < 0 \) and \( |H(x^*)| > 0 \). (These two inequalities imply that \( f_{22}''(x^*) < 0 \).) Thus the extra condition, in addition to the two conditions
Similarly, a sufficient condition for a stationary point $x^*$ of a function of two variables to be a local maximizer are $f_{11}''(x^*) > 0$ and $\left| H(x^*) \right| > 0$ (which imply that $f_{22}''(x^*) > 0$).

In particular, if, for a function of two variables, $\left| H(x^*) \right| < 0$, then $x^*$ is neither a local maximizer nor a local minimizer. (Note that this condition is only sufficient, not necessary.)

A stationary point that is neither a local maximizer nor a local minimizer is called a saddle point. Examples are the point $(0, 0)$ for the function $f(x, y) = x^3 - y^3$ and the point $(0, 0)$ for the function $f(x, y) = x^4 - y^4$. In both cases, $(0, 0)$ is a maximizer in the $y$ direction given $x = 0$ and a minimizer in the $x$ direction given $y = 0$; the graph of each function resembles a saddle for a horse. Note that not all saddle points look like saddles. For example, every point $(0, y)$ is a saddle point of the function $f(x, y) = x^3$. From the results above, a sufficient, though not necessary, condition for a stationary point $x^*$ of a function $f$ of two variables to be a saddle point is $\left| H(x^*) \right| < 0$.

(Note that the book *Mathematics for Economists* by Simon and Blume (p. 399) defines a saddle point to be a stationary point at which the Hessian is indefinite. Under this definition, $(0, 0)$ is a saddle point of $f(x, y) = x^3 - y^3$, but $(0, 0)$ is not a saddle point of $f(x, y) = x^4 - y^4$. The definition I give appears to be more standard.)

**Example**

Consider the function $f(x, y) = x^3 + y^3 - 3xy$. The first-order conditions for an optimum are

\[
3x^2 - 3y = 0 \\
3y^2 - 3x = 0.
\]

Thus the stationary points satisfy $y = x^2 = y^3$, so that either $(x, y) = (0, 0)$ or $y = 1$. So there are two stationary points: $(0, 0)$, and $(1, 1)$.

Now, the Hessian of $f$ at any point $(x, y)$ is

\[
H(x, y) = \begin{bmatrix}
6x & -3 \\
-3 & 6y
\end{bmatrix}.
\]

Thus $\left| H(0, 0) \right| = -9$, so that $(0, 0)$ is neither a local maximizer nor a local minimizer (i.e. is a saddle point). We have $f_{11}''(1, 1) = 6 > 0$ and $\left| H(1, 1) \right| = 36 - 9 > 0$, so that $(1, 1)$ is a local minimizer.
Example

Consider the function \( f(x, y) = 8x^3 + 2xy - 3x^2 + y^2 + 1 \). We have
\[
\begin{align*}
f'_x(x, y) &= 24x^2 + 2y - 6x \\
f'_y(x, y) &= 2x + 2y.
\end{align*}
\]
So the Hessian is
\[
\begin{pmatrix}
48x - 6 & 2 \\
2 & 2
\end{pmatrix}
\]
To find the stationary points of the function, solve the first-order conditions. From the second equation have \( y = -x \); substituting this into first equation we find that \( 24x^2 - 8x = 8x(3x - 1) = 0 \). This equation has two solutions, \( x = 0 \) and \( x = 1/3 \). Thus there are two solutions of the first-order conditions:

\((x^*, y^*) = (0, 0)\) and \((x^{**}, y^{**}) = (1/3, -1/3)\).

Now look at the second-order condition. We have
\[
\begin{align*}
f''_{xx}(x, y) &= 48x - 6, \\
f''_{yy}(x, y) &= 2, \quad \text{and} \\
f''_{xy}(x, y) &= 2.
\end{align*}
\]
Now look at each stationary point in turn:

\((x^*, y^*) = (0, 0)\)
We have \( f''_{xx}(0, 0) = -6 < 0 \) and  
\[
f''_{xx}(0, 0) f''_{yy}(0, 0) - (f''_{xy}(0, 0))^2 = -16 < 0.
\]
So \((x^*, y^*) = (0, 0)\) is neither a local maximizer nor a local minimizer (i.e. it is a saddle point).

\((x^{**}, y^{**}) = (1/3, -1/3)\)
We have \( f''_{xx}(1/3, -1/3) = 10 > 0 \) and  
\[
f''_{xx}(1/3, -1/3) f''_{yy}(1/3, -1/3) - (f''_{xy}(1/3, -1/3))^2 = 96/3 - 16 = 16 > 0.
\]
So \((x^{**}, y^{**}) = (1/3, -1/3)\) is a local minimizer. The minimum value of the function is \( f(1/3, -1/3) = 23/27 \).

5.2 Exercises on local optima

1. Is \( x = 1 \) a local maximum or a local minimum of the function \( f(x) = -x^3 + 3x - 2 \), or neither?
2. Find all the local maxima and minima (if any) of the following functions.
   a. \( f(x, y) = -x^3 + xy - y^2 + 2x + y \)
   b. \( f(x, y) = e^x - 2x + 2y^3 + 3 \)
3. Find all the local maxima and minima and all the global maxima and minima, if any, of the following functions. For any extreme point that you find, give both the
maximizer or minimizer (value of $x$) and the maximum or minimum (value of $f(x)$).

a. $f(x) = x^3 + 3$ on $[-1, 1]$

b. $f(x) = x^3 - 3x + 5$ on $[-3, 3]$

c. $f(x) = x + 1/x$ on $[1/2, 2]$

d. $f(x) = (x - 2)^3$ on $[0, 4]$.  

4. Find all the local maxima (if any) of the following functions. For each local maximum that you find, determine, if possible, whether it is a global maximum.

a. $f(x, y) = (1/3)x^3 + 2xy - 2y^2 - 6x.$

b. $f(x, y) = 3xy - x^3 - y^3.$

5. Let $f(x, y) = (y - x^2)(y - 2x^3).$

a. Draw a figure showing the regions of the $(x, y)$ plane at which this function has positive values and the regions at which it has negative values.

b. Fix a number $a$ and restrict attention to values of $x$ and $y$ for which $x = ay.$ That is, for each number $a,$ consider the function $g_a(y) = f(ay, y).$ Show that for every value of $a$ the point $y = 0$ is a local minimum of $g_a(y)$.

c. Is the point $(x, y) = (0,0)$ a local minimum of $f$?

6. Find all the local maxima and minima and all the global maxima and minima, if any, of the following functions. For any extreme point that you find, give both the maximizer or minimizer (value of $x$) and the maximum or minimum (value of $f(x)$).

a. $f(x) = 3x^3 - 5x^2 + x$ on the interval $[0, 1]$

b. $f(x) = 3x^3 - 5x^2 + x$ on the interval $[0, 2]$

c. $f(x) = (x - 4)^3 + 5$ on the interval $[-5, 5]$

7. Consider the function $f(x, y) = (x - 2)^3 + (y - 3)^3.$

a. Show that this function has a minimum at $(x, y) = (2, 3)$ (without using any calculus).

b. Find all the solutions of the first-order conditions.

c. Is the Hessian of $f$ positive definite at any solution of the first-order conditions?

8. Find all the local maximizers and minimizers of the following functions.

a. $f(x, x, x) = x_i^3 + 3x_i^2 - 3x_jx_i + 4x_kx_i + 6x_l^2.$

b. $f(x, x, x) = 29(x_i^3 + x_j^3 + x_k^3).$

c. $f(x, x, x) = x_i^3 + x_j^3 - x_i + x_j^2 + x_k^3 + 3x_l^3.$

5.2 Solutions to exercises on local optima

1. $f'(x) = -3x^2 + 3$ and $f''(x) = -6x$, so $f'(1) = 0$ and $f''(1) = -6 < 0.$ Thus $x = 1$ is a local maximizer.

2. 

a. $f'(x, y) = -2x + y + 2$ and $f'(x, y) = x - 2y + 1.$ So the first-order conditions have a unique solution, $(x, y) = (5/3, 4/3).$ We have $f''_{xx}(x, y) = -2$, $f''_{xy}(x, y) = -2$, and $f''_{yy}(x, y) = 1$, so $f''_{xx}(5/3, 4/3) f''_{xy}(5/3, 4/3) = f''_{xy}(5/3, 4/3) - f''_{xx}(5/3, 4/3) f''_{yy}(5/3, 4/3) = 4 > 0$.
(f''_{tx}(5/3, 4/3))^2 = 3 > 0. Hence the Hessian of $f$ is negative definite at $(5/3, 4/3)$, so that $(x, y) = (5/3, 4/3)$ is a local maximizer.

b. $f'(x, y) = 2e^2 - 2, f''_{tx}(x, y) = 4y$. So the first-order conditions have a unique solution, $(x, y) = (0, 0)$. We have $f''_{tx}(x, y) = 4e^2, f''_{tx}(x, y) = 4$, and $f''_{tx}(x, y) = 0$, so $f''_{tx}(0, 0)f''_{tx}(0, 0) - (f''_{tx}(0, 0))^2 = 16e^2 > 0$. Hence the Hessian of $f$ is negative definite at $(0, 0)$, so that $(x, y) = (0, 0)$ is a local minimizer.

3.

a. We have $f'(x) = 2x$, so the function has a single stationary point, $x = 0$. We have also $f''(x) = 2 > 0$, so $x = 0$ is a local minimizer. We have also $f(-1) = 4$ and $f(1) = 4$, so that $x = 0$ is the only global minimizer (value 3), and $x = -1$ and $x = 1$ are both global maximizers (value 4).

b. We have $f'(x) = 3x^2 - 3$, so the function has two stationary points, $x = 1$ and $x = -1$. We have $f''(x) = 6x$, so that $f''(1) > 0$ and $f''(-1) < 0$. Thus $x = 1$ is a local minimizer (minimum value 3) and $x = -1$ is a local maximizer (maximum value 7). We have also $f(-3) = -13$ and $f(3) = 23$, so $x = -3$ is the only global minimizer (value −13) and $x = 3$ is the only global maximizer (value 23).

c. We have $f'(x) = 1 - 1/x^2$, so the function has a single stationary point in the interval $[1/2, 2]$, namely $x = 1$. We have $f''(x) = 2/x^2$, so $x = 1$ is a local minimizer (value 2). We have also $f(1/2) = 5/2$ and $f(2) = 5/2$, so $x = 1$ is the only global minimizer (value 2) and $x = 1/2$ and $x = 2$ are both global maximizers (value 5/2).

d. We have $f'(x) = 6(x - 2)^2$, so the function has a single stationary point, $x = 2$. We have $f''(x) = 30(x - 2)^2$, so that $f''(2) = 0$. Thus this information is not enough to tell us whether $x = 2$ is a local maximizer or minimizer, or neither. We have also $f(0) = 64$ and $f(4) = 64$, so $x = 2$ is the only global minimizer (value 0), and $x = 0$ and $x = 4$ are both global maximizers (value 64). We conclude that $x = 2$ is the only local minimizer, and $x = 0$ and $x = 4$ are both local maximizers.

4.

a. The first-order conditions are

\[
\begin{align*}
x^2 + 2y - 6 &= 0 \\
2x - 4y &= 0.
\end{align*}
\]

b. From the second equation we have $x = 2y$, so from the first equation we have

c. $4y^2 + 2y - 6 = 0$,
d. or
e. $2y^2 + y - 3 = 0$,
f. or
g. $(2y + 3)(y - 1) = 0$.
h. Thus $y = 1$ or $y = -3/2$. Hence the equations have two solutions: $(2, 1)$ and $(-3, -3/2)$. 

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i. The Hessian of the function is
\[
\begin{pmatrix}
2x & 2 \\
2 & 0
\end{pmatrix}
\]

j. For \(x = 2\) this matrix is indefinite; \(f''(2, 1) > 0\) and \(f''(2, 1) < 0\), so \((2, 1)\) is a saddle point. For \(x = -3\) the matrix is negative definite, so that \((-3, -3/2)\) is a local maximizer.

k. The point \((-3, -3/2)\) is not a global maximizer, because for \(y = 0\) and \(x\) arbitrarily large, the value of the function is arbitrarily large.

l. The first-order conditions are
\[
\begin{align*}
3y - 3x^2 &= 0 \\
3x - 3y^2 &= 0.
\end{align*}
\]

m. These equations have two solutions, \((0, 0)\) and \((1, 1)\).

n. The Hessian of the function is
\[
\begin{pmatrix}
-6x & 3 \\
3 & 0
\end{pmatrix}
\]

o. At \((1,1)\) this Hessian is negative definite, while at \((0,0)\) it is indefinite.

p. Thus \((1,1)\) is a local maximizer. It is not a global maximizer, because \(f(1, 1) = 1\), while \(f(-1, -1) = 5\) (for example).

5.

a. In the figure below, the function is positive in the red region (i.e. when \(x > y^2\) and \(x > 2y^2\), or \(x < y^2\) and \(y < 2y^2\)) and negative in the green region.
b. We have \( g(y) = f(ay, y) = (y - ay^2)(y - 2ay^2) = y^3 - 3ayy^2 + 2a^2y^4 \). The derivative is \( 2y - 9ay^2 + 8a^2y^3 \), which is zero when \( y = 0 \). The second derivative is \( 2 - 18a^2y + 24a^4y^2 \), which is positive at \( y = 0 \), so for any value of \( a \) the function has local minimum at \( 0 \).

c. No: look at the diagram for part (a). Every disk centered at \((0, 0)\), not matter how small, contains points at which the function is negative (points between the two parabolas).

6.

a. First find the stationary points: these are the solutions of \( f'(x) = 0 \), or \( 9x^2 - 10x + 1 = 0 \). Thus the stationary points are \( x = 1/9 \) and \( x = 1 \). The values of the function at these points are \( f(1/9) = 3(1/9)^3 - 5(1/9)^2 + 1/9 = (1/243)(1 - 15 + 27) = 13/243 \) and \( f(1) = -1 \).

Now find the values of the function at the endpoints of the interval: \( f(0) = 0 \) and \( f(1) = -1 \).

Thus the (global) maximizer of the function on the interval \([0, 1]\) is \( x = 1/9 \) and the (global) minimizer is \( x = 1 \). (There are no other local maximizers or minimizers.)

b. As in the previous part, the stationary points are \( x = 1/9 \) and \( x = 1 \), with \( f(1/9) = 13/243 \) and \( f(1) = -1 \).

The values of the function at the endpoints of the interval are \( f(0) = 0 \) and \( f(2) = 6 \).

Thus the (global) maximizer of the function on the interval \([0, 2]\) is \( x = 2 \) and the (global) minimizer is \( x = 1 \).

The only other stationary point is \( x = 1/9 \). We have \( f''(x) = 18x - 10 \), so that \( f''(1/9) < 0 \). Thus \( x = 1/9 \) is a local maximizer.

c. We have \( f'(x) = 2(x - 4) \) and hence \( f''(x) = 2 \) for all values of \( x \). Thus the function \( f \) is convex. Hence any stationary point is a global minimizer.

There is a single stationary point, \( x = 4 \). Thus the function has a single global minimizer, \( x = 4 \).

We have \( f(-5) = 86 \) and \( f(5) = 6 \), so the global maximizer is \( x = -5 \).

7.

a. \( f(2, 3) = 0 \leq (x - 2)^0 + (y - 3)^0 \) for any \( x \) and \( y \), because \( z^0 \geq 0 \) for all \( z \).

b. \( f_1'(x, y) = 4(x - 2)^2, f_2'(x, y) = 4(y - 3)^2 \). So the only solution of first-order conditions is \( (x, y) = (2, 3) \).
c. \( f_{11}''(x, y) = 12(x - 2)^2, \quad f_{22}''(x, y) = 12(y - 3)^2, \quad \text{and} \quad f_{12}''(x, y) = 0, \) so that we have \( f_{11}''(2, 3) \cdot f_{22}''(2, 3) - (f_{12}''(2, 3))^2 = 0. \) Thus the Hessian is not positive definite at any solution of the first-order conditions. (Hence we cannot conclude from the second-order conditions that \((2, 3)\) is a minimizer of the function, though we know that it is.)

8.

a. First-order conditions:

\[
\begin{align*}
f_{1}'(x_1, x_2, x_3) &= 2x_1 - 3x_2 = 0 \\
f_{2}'(x_1, x_2, x_3) &= -3x_1 + 6x_2 + 4x_3 = 0 \\
f_{3}'(x_1, x_2, x_3) &= 4x_2 + 12x_3 = 0
\end{align*}
\]

b. These conditions have a unique solution, \((x_1, x_2, x_3) = (0, 0, 0).\) The Hessian matrix is

\[
\begin{pmatrix}
2 & -3 & 0 \\
-3 & 6 & 4 \\
0 & 4 & 12
\end{pmatrix}
\]

c. The leading principal minors are \(2 > 0, 3 > 0,\) and \((2)(72-16)-(3)(-36) = 4 > 0,\) so that \((x_1, x_2, x_3) = (0,0,0)\) is a local minimizer. (The function is convex, so this minimum is in fact a global minimizer.)

d. First-order conditions: \( f_{i}'(x_1, x_2, x_3) = -2x_i = 0 \) for \(i = 1, 2, 3.\) There is a unique solution, \((x_1, x_2, x_3) = (0,0,0).\) The Hessian matrix is

\[
\begin{pmatrix}
-2 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & -2
\end{pmatrix}
\]

e. The leading principal minors are \(-2 < 0, 4 > 0,\) and \((-2)(4) = -8 < 0,\) so that \((x_1, x_2, x_3) = (0,0,0)\) is a local maximizer.

f. First-order conditions:

\[
\begin{align*}
f_{1}'(x_1, x_2, x_3) &= 2x_1 + x_3 = 0 \\
f_{2}'(x_1, x_2, x_3) &= 2x_1 + x_3 - 1 = 0 \\
f_{3}'(x_1, x_2, x_3) &= x_1 + x_2 + 6x_3 = 0
\end{align*}
\]

g. These conditions have a unique solution, \((x_1, x_2, x_3) = (1/20, 11/20, -2/20).\) The Hessian matrix is

\[
\begin{pmatrix}
2 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
\begin{align*}
0 & \quad 2 & \quad 1 \\
1 & \quad 1 & \quad 6
\end{align*}

h. The leading principal minors are $2 > 0$, $4 > 0$, and $(2)(11) + (1)(-2) = 20 > 0$, so that $(x_1, x_2, x_3) = (1/20, 11/20, -2/20)$ is a local minimizer.

### 5.3 Conditions under which a stationary point is a global optimum

#### One variable

Let $f$ be a differentiable concave function. Then for every point $x$, no point on the graph of $f$ lies above the tangent to $f$ at $x$. Thus if $f'(x) = 0$ (i.e. if $x$ is a stationary point of $f$), the point $x$ is a (global) maximizer of $f$. Similarly, a differentiable convex function lies nowhere below any of its tangents. Thus we have the following result.

**Proposition**

Let $f$ be a differentiable function defined on the interval $I$, and let $x$ be in the interior of $I$. Then

- if $f$ is concave then $x$ is a global maximizer of $f$ in $I$ if and only if $x$ is a stationary point of $f$
- if $f$ is convex then $x$ is a global minimizer of $f$ in $I$ if and only if $x$ is a stationary point of $f$.

Now, a twice-differentiable function is concave if and only if its second derivative is nonpositive (and similarly for a convex function), so we deduce that if $f$ is a twice-differentiable function defined on the interval $I$ and $x$ is in the interior of $I$, then

- $f''(z) \leq 0$ for all $z \in I \Rightarrow [x$ is a global maximizer of $f$ in $I$ if and only if $f'(x) = 0]$.
- $f''(z) \geq 0$ for all $z \in I \Rightarrow [x$ is a global minimizer of $f$ in $I$ if and only if $f'(x) = 0]$.

**Example**

Consider the problem $\max x - x^2$ subject to $x \in [-1,1]$. The function $f$ is concave; its unique stationary point is 0. Thus its global maximizer is 0.

**Example**

A competitive firm pays $w$ for each unit of an input. It obtains the revenue $p$ for each unit of output that it sells. Its output from $x$ units of the input is $\sqrt{x}$. For what value of $x$ is its profit maximized?

The firm's profit is $p\sqrt{x} - wx$. The derivative of this function is $(1/2)p\sqrt{x}^2 - w$, and the second derivative is $-(1/4)p\sqrt{x}^2 \leq 0$, so the function is concave. So the global
maximum of the function occurs at the stationary point. Hence the maximizer solves \((1/2)px^{1/2} - w = 0\), so that \(x = (p/2w)^2\).

What happens if the production function is \(x^3\)?

### Many variables

A differentiable concave function of many variables always lies below, or on, its tangent plane, and a differentiable convex function always lies above, or on, its tangent plane. Thus as for functions of a single variable, every stationary point of a concave function of many variables is a maximizer, and every stationary point of a convex function of many variables is a minimizer, as the following result claims.

**Proposition**

Suppose that the function \(f\) has continuous partial derivatives in a convex set \(S\) and let \(x\) be in the interior of \(S\). Then

- if \(f\) is concave then \(x\) is a global maximizer of \(f\) in \(S\) if and only if it is a stationary point of \(f\)
- if \(f\) is convex then \(x\) is a global minimizer of \(f\) in \(S\) if and only if it is a stationary point of \(f\).

As in the case of functions of a single variable, we can combine this result with a previous result characterizing twice-differentiable concave and convex functions to conclude that if \(f\) is a function with continuous partial derivatives of first and second order on a convex set \(S\), and \(x\) is in the interior of \(S\), then

- \(H(z)\) is negative semidefinite for all \(z \in S\) \(\Rightarrow\) \(x\) is a global maximizer of \(f\) in \(S\) if and only if \(x\) is a stationary point of \(f\)
- \(H(z)\) is positive semidefinite for all \(z \in S\) \(\Rightarrow\) \(x\) is a global minimizer of \(f\) in \(S\) if and only if \(x\) is a stationary point of \(f\),

where \(H(x)\) denotes the Hessian of \(f\) at \(x\).

Be sure to notice the difference between the form of this result and that of the result on local optima. To state briefly the results for maximizers together:

**Sufficient conditions for local maximizer**: if \(x^*\) is a stationary point of \(f\) and the Hessian of \(f\) is negative definite at \(x^*\) then \(x^*\) is a local maximizer of \(f\)

**Sufficient conditions for global maximizer**: if \(x^*\) is a stationary point of \(f\) and the Hessian of \(f\) is negative semidefinite for all values of \(x\) then \(x^*\) is a global maximizer of \(f\).

**Example**

Consider the function \(f(x, y) = x^3 + xy + 2y^2 + 3\), defined on the domain \((−∞, ∞)\). We have

\[f'(x, y) = 2x + y\]
\[ f'(x, y) = x + 4y \]

so that the function has a single stationary point, \((x, y) = (0, 0)\).

We have

\[ f''(x, y) = 2, \ f'''(x, y) = 4, \text{ and } f''''(x, y) = 1, \]

so the Hessian of \( f \) is

\[
\begin{pmatrix}
2 & 1 \\
1 & 4
\end{pmatrix}
\]

which is positive definite for all values of \((x, y)\). (The matrix is independent of \((x, y)\).) Hence \( f \) is convex (in fact, strictly convex).

Thus the global minimizer of the function is \((0, 0)\); the minimum value of the function is 3.

**Example**

Consider the function \( f(x, y) = x^4 + 2y^2 \), defined on the domain \((-\infty, \infty)\). We have

\[
\begin{align*}
f'(x, y) &= 4x^3, \\
f'(x, y) &= 4y,
\end{align*}
\]

so that the function has a single stationary point, \((x, y) = (0, 0)\). We have

\[ f''(x, y) = 12x^2, \ f'''(x, y) = 4, \text{ and } f''''(x, y) = 0, \]

so the Hessian of \( f \) is

\[
\begin{pmatrix}
12x^2 & 0 \\
0 & 4
\end{pmatrix}
\]

This matrix is positive semidefinite for all values of \((x, y)\), so that \((0, 0)\) is the unique global minimizer of the function.

**Example**

Consider the function \( f(x, y) = x^3 + 2y^2 \), defined on the domain \((-\infty, \infty)\). We have

\[
\begin{align*}
f'(x, y) &= 3x^2, \\
f'(x, y) &= 4y,
\end{align*}
\]

so that the function has a single stationary point, \((x, y) = (0, 0)\). We have

\[ f''(x, y) = 6x, \ f'''(x, y) = 4, \text{ and } f''''(x, y) = 0, \]
so the Hessian of \( f \) is
\[
\begin{pmatrix}
6x & 0 \\
0 & 4
\end{pmatrix}
\]

At \((x, y) = (0, 0)\) this matrix is positive semidefinite, but not positive definite. Thus we cannot tell from this analysis whether \((0, 0)\) is a local maximizer or local minimizer, or neither. (In fact, it is neither: for all \(\varepsilon > 0\) the function is positive at \((\varepsilon, 0)\) and negative at \((-\varepsilon, 0)\).

At other values of \((x, y)\) the matrix is not positive semidefinite (for example, if \(x < 0\) the matrix is not positive semidefinite), so the function is not concave. Thus we cannot conclude that \((0, 0)\) is either a global maximizer or minimizer.

In summary, we can conclude from this analysis only that if the function has a minimizer, then this minimizer is \((0, 0)\), and if it has a maximizer, then this maximizer is \((0, 0)\). In fact, the function does not have either a minimizer or maximizer---it is an arbitrarily large negative number when \(x\) is an arbitrarily large negative number, and is an arbitrarily large positive number when \(y\) is an arbitrarily large positive number.

**Example**
Consider the function \( f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 \), found to be strictly convex in a previous example. It has a unique stationary point, \((x_1, x_2, x_3) = (0, 0, 0)\). Hence its unique global minimizer is \((0, 0, 0)\), with a value of 0.

**Example**
Consider a firm with the production function \( f \), defined over vectors \((x_1, \ldots, x_n)\) of inputs. Assume that \( f \) is concave. The firm's profit function is
\[
\pi(x_1, \ldots, x_n) = pf(x_1, \ldots, x_n) - \sum_{j=1}^n w_j x_j
\]
where \( p \) is the price of the output of the firm and \( w_j \) is the price of the \( j \)th input. The second term in the profit function is linear, hence concave, and so the function \( \pi \) is concave (by a previous result). So the input bundle \((x_1^*, \ldots, x_n^*) > 0\) maximizes the firm's profit if and only if
\[
p f'(x^*_j) = w_j
\]
(i.e. the value of the marginal product of each input \( j \) is equal to its price).

### 5.3 Exercises on conditions under which a stationary point is a global optimum

1. Find all the local maxima and minima (if any) of the following functions, and determine whether each local extremum is a global extremum.
   a. \( f(x, y) = -x^4 + 2xy + y^2 + x \).
   b. \( f(x, y, z) = 1 - x^2 - y^2 - z^2 \).
2. A competitive firm receives a price \( p > 0 \) for each unit of its output, pays a price \( w > 0 \) for each unit of its single input. Its output from using \( x \) units of the variable input is \( f(x) \). What do the results in Section 4 tell us about the value(s) of \( x \) that maximize the firm's profit in each of the following cases? Find the firm's maximum profit in each case.
   a. \( f(x) = x^a \). [Refer to a previous exercise.]
   b. \( f(x) = x \). [Treat the cases \( p = w \) and \( p \neq w \) separately.]
   c. \( f(x) = x^a \).

3. Find all the local maximizers and local minimizers (if any) of the function \( f(x, y) = e^{3x} - 3x + 4y - 1 \) (without any constraints). Determine whether each local maximizer you find is a global maximizer and whether each local minimizer is a global minimizer.

4. Let \( f(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2 + 3x_1 - 2x_2 + 1 \).
   a. Is \( f \) convex, concave, or neither?
   b. Find any local maxima or minima of \( f \). Are they global maxima/minima?

5. Using the results that relate the solutions of the first-order conditions with the solutions of maximization/minimization problems, what can you say about the local and global maxima and minima of the following functions?
   a. (10) \( f(x, y, z) = x^2 + 3y^2 + 6z^2 - 3xy + 4yz \).
   b. (10) \( f(x, y) = -x^2 - y^2 + 3xy \).

6. Suppose that a firm that uses 2 inputs has the production function \( f(x_1, x_2) = 12x_1^{1/3}x_2^{1/2} \) and faces the input prices \( (p_1, p_2) \) and the output price \( q \).
   a. Show that \( f \) is concave, so that the firm's profit is concave.
   b. Find a global maximum of the firm's profit (and give the input combination that achieves this maximum).

7. Suppose that a firm that uses 2 inputs has the production function \( f(x_1, x_2) = 4x_1^{1/3}x_2^{1/4} \) (where \( (x_1, x_2) \) is the pair of amounts of the inputs) and faces the input prices \( (p_1, p_2) \) and the output price \( q \).
   a. Show that the firm's profit is a concave function of \( (x_1, x_2) \). (To show that \( f \) is concave, use first principles, not a general result about the concavity of Cobb-Douglas production functions.)
   b. Find the pair of inputs that maximizes the firm's profit.

8. Solve the problem
   \[
   \max \int_0^1 - (x-y)g(y)dy,
   \]
   where \( g \) is a function for which \( 0 \leq g(y) \leq 1 \) for all \( y \) and \( \int_0^1 g(y)dy = 1 \).

9. A firm produces output using \( n \) inputs according to the continuous production function \( f \). It sells output at the price \( p \) and pays \( w_i \) per unit for input \( i, i = 1, ..., n \). Consider its profit maximization problem
   \[
   \max pf(z) - w \cdot z \text{ subject to } z \geq 0.
   \]
   Throughout the problem fix the value of \( w \).
   a. Does the extreme value theorem imply that this problem has a solution?
b. Denote by \( \pi(p) \) the firm's maximal profit, given \( p \). Show that \( \pi'(p) = f(z^*(p)) \). [Hint: Use the first-order conditions for the maximization problem.]

c. Let \( q(p) = f(z^*(p)) \), the output that the firm produces. By a previous problem, \( \pi \) is convex. What testable restrictions does this property of \( \pi \) imply for \( q \), given the result in (b)? (What does the convexity of \( \pi \) tell you about its second derivative? How is the second derivative related to the derivative of \( q^* \)?)

10. A firm produces the output of a single good in two plants, using a single input.

The amount of the input it uses in plant \( i \) is \( z_i \). The output in each plant depends on the inputs in both plants (there is an interaction between the plants): the output in plant 1 is \( f(z_1, z_2) \) and that in plant 2 is \( g(z_1, z_2) \), where \( f \) and \( g \) are differentiable.

The price of output is \( p > 0 \) and the price of the input is \( w > 0 \). Thus the firm's profit, which it wishes to maximize, is \( p[f(z_1, z_2) + g(z_1, z_2)] - w(z_1 + z_2) \).

a. What are the first-order conditions for a solution of the firm's problem?

b. If \( (z_i^*, z_2^*) \) maximizes the firm's profit and \( z_i^* > 0 \) for \( i = 1, 2 \), then does \( (z_i^*, z_2^*) \) necessarily satisfy the first-order conditions?

c. If \( (z_i^*, z_2^*) \) satisfies the first-order conditions and \( z_i^* > 0 \) for \( i = 1, 2 \), then does \( (z_i^*, z_2^*) \) necessarily maximize the firm's profit? If not, are there any conditions on \( f \) and \( g \) under which \( (z_i^*, z_2^*) \) does necessarily maximize the firm's profit?

11. Find all the local maxima and minima (if any) of the following functions. Are any of these maxima and minima global?

a. \( f(x, y) = -x^4 + 2xy + y^2 + x \)

b. \( f(x, y, z) = 1 - x^2 - y^2 - z^2 \).

12. You have the following information about \( f \), a function of \( n \) variables:

a. \( f \) is differentiable

b. \( f'(x) = 0 \) for \( i = 1, ..., n \) if and only if \( x = x_1, x = x_2, x = x_3, \ldots, x = x_i \), or \( x = x_i \).

c. the Hessian \( H \) of \( f \) is negative definite at \( x_i \), negative semidefinite at \( x_i \), positive definite at \( x_i \), negative definite at \( x_n \), and positive definite at \( x_n \).

d. \( f(x_1) = 3, f(x_2) = 6, f(x_3) = 2, f(x_4) = 6, f(x_5) = 5 \).

What do you know about the local and global maximizer(s) and minimizers of \( f \) (in the absence of any constraint on \( x \))?

o A firm faces uncertain demand \( D \) and has inventory \( I \). It chooses its stock level \( Q \geq 0 \) to minimize

\[
g(Q) = c(Q - I) + h\int_0^Q f(D - D) f(D) dD + aQ f(Q - D) f(D) dD,
\]

where \( c, I, h, p \), and \( a \) are positive constants with \( p > c \), and \( f \) is a nonnegative function that satisfies \( \int_{Q}^{f}(D) dD = 1 \) (so that it can be interpreted as a probability distribution function).

a. Denote by \( Q^* \) the stock level that minimizes \( g(Q) \). Write down the first-order condition for \( Q^* \) (in terms of \( c, I, h, p \), and integrals involving the function \( f \)).
b. What is the relation (if any) between an interior solution of the first-order condition and the solution of the problem? (Be precise.)
c. Find the solution $Q^*$ in the case that $f(D) = 1/a$ for all $D$.

-1. A politician chooses the number of hours $h \geq 0$ of advertising that she buys; the cost of $h$ hours is $c(h)$, where $c$ is a convex function. She wishes to maximize $g(\pi(h)) - c(h)$, where $\pi(h)$ is her probability of winning when she buys $h$ hours of advertising. Assume that $\pi$ is a concave function, and $g$ is an increasing and concave function.

   a. Write down the first-order condition for an interior solution (i.e. a solution $h^*$ with $h^* > 0$) of this problem.
   b. What is the relation (if any) between an interior solution of the first-order condition and the solution of the problem? (Be precise.)

-1. A firm sells goods in two markets. In each market $i$ the price is fixed at $p_i$, with $p_i > 0$. The amount of the good the firm sells in each market depends on its advertising expenditure in both markets. If it spends $a_i$ on advertising in market $i$, for $i = 1, 2$, its sales in market 1 are $f(a_1, a_2)$ and its sales in market 2 are $g(a_1, a_2)$, where $f$ and $g$ are twice differentiable functions. The firm's cost of production is zero. Thus its profit is

$$p_i f(a_1, a_2) + p_2 g(a_1, a_2) - a_1 - a_2.$$ It chooses $(a_1, a_2)$ to maximize this profit.

   a. What are the first-order conditions for the firm's optimization problem?
   b. Suppose that $(a_1^*, a_2^*)$ maximizes the firm's profit and $a_i^* > 0$ for $i = 1, 2$. Does $(a_1^*, a_2^*)$ necessarily satisfy the first-order conditions?
   c. Suppose that $(a_1^*, a_2^*)$ satisfies the first-order conditions and $a_i^* > 0$ for $i = 1, 2$. Under what condition is $(a_1^*, a_2^*)$ a local maximizer of the firm's profit?
   d. Suppose that $(a_1^*, a_2^*)$ satisfies the first-order conditions and $a_i^* > 0$ for $i = 1, 2$. Under what conditions on the functions $f$ and $g$ is $(a_1^*, a_2^*)$ necessarily a global maximizer of the firm's profit?

### 5.3 Solutions to exercises on conditions under which a stationary point is a global optimum

1. a. First-order conditions are

$$f'_x(x, y) = -3x^2 + 2y + 1 = 0$$
$$f'_y(x, y) = 2x + 2y = 0$$

b. From the second we have $y = -x$, so that the first condition is $3x^2 + 2x - 1 = 0$, which has solutions $x = 1/3$, $x = -1$. Thus there are two solutions of the first-order conditions: $(1/3, -1/3)$ and $(-1, 1)$.

c. The Hessian of $f$ is
d. which is indefinite for $x = 1/3$, with $f''(1/3, -1/3) < 0$ and $f''_{11}(1/3, -1/3) > 0$, and positive definite for $x = -1$. Thus $(1/3, -1/3)$ is a saddle point and $(-1, 1)$ is a local minimizer.

e. $(-1, 1)$ is not a global minimizer: for example, $f(2,0) = -6 < f(-1, 1) = -1$.

f. First-order conditions: $f_1'(x, y, z) = -2x = 0$, $f_2'(x, y, z) = -2y = 0$, $f_3'(x, y, z) = -2z = 0$. There is a unique solution, $(x, y, z) = (0, 0, 0)$. The Hessian matrix is

$$
\begin{pmatrix}
-2 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & -2
\end{pmatrix}
$$

g. The leading principal minors are $-2 < 0, 4 > 0$, and $(-2)(4) = -8 < 0$, so that $(x, x, x) = (0, 0, 0)$ is a local maximizer; the value at this maximum is 1.

h. The Hessian is negative definite for all $(x, y, z)$, so that $f$ is concave. Thus $(0, 0, 0)$ is a global maximizer.

2.

a. The firm's profit, as a function of $x$, is $px^{1/4} - wx$. By the previous exercise mentioned in the problem, this function is concave. Its stationary points are the solutions of $p(1/4)x^{-3/4} - w = 0$, so it has a single stationary point, $x = (p/4w)^{4/3}$. This value of $x$ is positive, so that it is in the interior of the interval on which $f$ is defined, and hence by the first result in this section is the only positive global maximizer of the firm's profit. The firm's profit when $x = (p/4w)^{4/3}$ is $p(p/4w)^{4/3} - w(p/4w)^{4/3} = (1/4^{4/3} - 1/4^{4/3})p^{4/3}/w^{4/3} = (3/4^{4/3})(p^{4/3}/w^{4/3}) > 0$. The firm's profit when $x = 0$ is 0, so the only maximizer of the firm's profit is $x = (p/4w)^{4/3}$.

b. If $f(x) = x$ then the firm's profit is $px - wx = (p - w)x$, which is concave (the second derivative is zero).

- If $p = w$ then every value of $x$ is a stationary point, so that by the first result in this section every value of $x > 0$ is a global maximizer of the firm's profit. The firm's profit at every such value of $x$ is 0, so that $x = 0$ is also a global maximizer.

- If $p \neq w$ the function has no stationary points. Thus the first result in the section implies that no value of $x > 0$ is a global maximizer. The firm's profit when $x = 0$ is 0, so this value of $x$ maximizes the firm's profit if $p < w$ (in which case the firm's profit for $x > 0$ is negative). If $p > w$, no value of $x$ maximizes the firm's profit, which increases without bound as $x$ increases.
c. If \( f(x) = x^2 \), the firm's profit is \( px^2 - wx \), which is not concave. By the first result in Section 4.4, if \( x \) maximizes profit then \( f'(x) = 0 \), or \( x = w/2p \). The firm's profit at this value of \( x \) is \( -w^2/4p < 0 \), while its profit for \( x = 0 \) is 0. Thus the only candidate for a maximizer of the firm's profit is \( x = 0 \). But looking at the function \( px^2 - wx \), we see that the firm's profit increases without bound, and hence has no maximum.

3. The first-order conditions are

\[
\begin{align*}
    f'(x, y) &= 3e^{3x} - 3 = 0 \\
    f'(x, y) &= 8y = 0.
\end{align*}
\]

4. These equations have a single solution, \((x, y) = (0, 0)\). The Hessian of the function is

\[
\begin{pmatrix}
    9e^{3x} & 0 \\
    0 & 8
\end{pmatrix}
\]

5. which is positive definite for all \((x, y)\). Thus the function is convex. Hence \((x, y) = (0, 0)\) is the unique global (and in particular local) minimizer. The function has no maximizer.

6. a. The Hessian matrix of \( f \) is

\[
\begin{pmatrix}
    2 & -1 \\
    -1 & 0
\end{pmatrix}
\]

b. This is positive definite, so \( f \) is convex.

c. The first-order conditions are

\[
\begin{align*}
    f_1'(x_1, x_2) &= 2x_1 - x_2 + 3 = 0 \\
    f_2'(x_1, x_2) &= -x_1 + 2x_2 - 2 = 0
\end{align*}
\]

d. These equations have a unique solution, \((x_1, x_2) = (-4/3, 1/3)\). Since \( f \) is convex, this is the global minimizer of \( f \) (and there are no other maximizers or minimizers).

7. a. The first-order conditions are

\[
\begin{align*}
    2x - 3y &= 0 \\
    6y - 3x + 4z &= 0 \\
    12z + 4y &= 0
\end{align*}
\]

b. The unique solution is \((x, y, z) = (0, 0, 0)\). The Hessian is
\[
\begin{pmatrix}
2 & -3 & 0 \\
-3 & 6 & 4 \\
0 & 4 & 12
\end{pmatrix}
\]

c. The leading principal minors of which are 2, 3, and 4. Thus the function is convex (in fact, strictly convex). Thus (0,0,0) is the global minimizer.
d. The first-order conditions are
\begin{align*}
-3x^2 + 3y &= 0 \\
-3y^2 + 3x &= 0
\end{align*}
e. These equations have two solutions, (0,0) and (1,1). The Hessian is
\[
\begin{pmatrix}
-6x & 3 \\
3 & -6x
\end{pmatrix}
\]
f. Thus the function is neither concave nor convex. At (0,0) the Hessian is neither positive nor negative semidefinite, so (0,0) is neither a maximizer nor a minimizer. At (1,1) the Hessian is negative definite, so that (1,1) is a local maximizer.

8.
a. The Hessian matrix at \((x_1, x_2)\) is
\[
\begin{pmatrix}
-(8/3)x_1^{-1/3}x_2^{1/2} & 2x_1^{-1/3}x_2^{-1/2} \\
-(8/3)x_1^{1/3}x_2^{-1/2} & 6x_1^{1/3}x_2^{1/2}
\end{pmatrix}
\]
b. The leading principal minors are \(-(8/3)x_1^{-1/3}x_2^{1/2} < 0\) and \(8x_1^{-1/3}x_2^{1/2} - 4x_1^{-1/3}x_2^{-1} = 4x_1^{-1/3}x_2^{-1} > 0\). Hence the Hessian is negative definite, so that \(f\) is concave.
c. The first-order conditions for a maximum of profit are
\begin{align*}
q f_1(x_1, x_2) - p_1 &= 4qx_1^{-1/3}x_2^{1/2} - p_1 = 0 \\
q f_2(x_1, x_2) &= 6qx_1^{1/3}x_2^{-1/2} - p_2 = 0
\end{align*}
d. These equations have a unique solution
\begin{align*}
x_1^* &= (24q/p_1)^3 \\
x_2^* &= (12q/p_2)^3
\end{align*}
e. Since the objective function is concave, this input combination is the one that globally maximizes the firm's profit. The value of the maximal profit is obtained by substituting these optimal input values into the profit function.
9. a. First consider the production function \( f \). The Hessian matrix at \((x_1, x_2)\) is
\[
\begin{pmatrix}
-(3/4)x_1^{-3/4}x_2^{1/4} & (1/4)x_1^{-3/4}x_2^{-3/4}
\end{pmatrix}
\]
b. The leading principal minors are \(-(3/4)x_1^{-3/4}x_2^{1/4} < 0\) and \((9/16)x_1^{-3/2}x_2^{-3/2} = 2x_1^{-3/2}x_2^{-3/2} > 0\). Hence the Hessian is negative definite, so that \( f \) is concave. Thus given that the firm’s revenue \( p_1x_1 + p_2x_2 \) is convex (and concave), the firm’s profit, \( qf(x_1, x_2) - p_1x_1 - p_2x_2 \) is concave.

c. The first-order conditions for a maximum of profit are
\[
q f_1'(x_1, x_2) - p_1 = qx_1^{-3/4}x_2^{1/4} - p_1 = 0
\]
\[
q f_2'(x_1, x_2) - p_2 = qx_1^{1/4}x_2^{-3/4} - p_2 = 0.
\]
d. These equations have a unique solution
\[
x_1^* = q^2/(p_1p_2)^{1/2}
\]
\[
x_2^* = q^2/(p_1p_2)^{1/2}
\]
e. (To obtain this solution you can, for example, isolate \( x_1 \) in the second equation and plug it into the first equation.)
f. The objective function is concave, so this input combination globally maximizes the firm’s profit.

10. The second derivative of the objective function is \(-2\), so that the objective function is concave. (Use Leibniz’s formula.) The unique solution of the first-order condition is \( x = \int yg(y)dy \), so this is the solution of the problem. (If \( g \) is a probability density function then \( \int yg(y)dy \) is the mean of \( y \).)

11. a. No, since the constraint set is not bounded.
b. We have
\[
\pi'(p) = f(z^*(p)) + \sum z_j f_j'(z^*(p))\partial z^*_j/\partial p - w_j\partial z^*_j/\partial p.
\]
Now, the first-order conditions for the firm’s maximization problem are \( pf_j'(z^*(p)) - w_j = 0 \) for all \( j \), so each term in square brackets is zero.

c. The convexity of \( \pi \) means that \( \pi''(p) \geq 0 \) for all \( p \). From (b) we have \( \pi''(p) = q'(p) \). Thus the theory predicts that the output of a profit-maximizing firm is an increasing function of the price of output. That is, the supply function is upward-sloping.

12. a. First-order conditions:
\[
p[f_j'(z_j^*, z_i^*) + g_j'(z_i^*, z_j^*)] - w = 0
\]
\[ p[f'(z_1^*, z_2^*) + g'(z_1^*, z_2^*)] - w = 0 \]

b. Yes: at an interior maximum the first-order conditions must be satisfied.
c. No: a solution of the first-order conditions is not necessarily a maximizer. If \( f \) and \( g \) are concave, however, then the objective function is concave, so the first-order conditions are sufficient.

13. a. First-order conditions are
\[
\begin{align*}
f'_1(x,y) &= -3x^2 + 2y + 1 = 0 \\
f'_2(x,y) &= 2x + 2y = 0
\end{align*}
\]
b. From the second we have \( y = -x \), so that the first condition is \( 3x^2 + 2x - 1 = 0 \), which has solutions \( x = (1/3), x = -1 \). Thus there are two solutions of the first-order conditions: \((1/3), -1/3)\) and \((-1,1)\).
c. The Hessian of \( f \) is
\[
\begin{pmatrix}
-6x & 2 \\
2 & 2
\end{pmatrix}
\]
d. which is indefinite for \( x = 1/3 \) and positive definite for \( x = -1 \). Thus \((1/3, -1/3)\) is a saddle point and \((-1,1)\) is a local minimizer. The point \((-1,1)\) is not a global minimizer since \( f(x,y) \to -\infty \) if \( x \to \infty \) when \( y = 0 \).
e. First-order conditions: \( f'(x,y,z) = -2x = 0 \), \( f'(x,y,z) = 2y = 0 \), \( f'(x,y,z) = -2z = 0 \). There is a unique solution, \((x, y, z) = (0,0,0)\). The Hessian matrix is
\[
\begin{pmatrix}
-2 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & -2
\end{pmatrix}
\]
f. The leading principal minors are \(-2 < 0\), \(4 > 0\), and \((-2)(4) = -8 < 0\), so that \((x, x, x, x) = (0,0,0)\) is a local maximizer; the value at this maximum is 1. Since the Hessian is negative semidefinite (in fact, negative definite) for all \( x \) (in fact, it doesn't depend on \( x \)), the function is concave, so that \((0,0,0)\) is a global maximizer.

14. The only possible local or global maximizers or minimizers of \( f \) are \( x_1, x_2, x_3, x_4, \) and \( x_5 \). Thus if \( f \) has a maximum then \( x_1 \) and \( x_4 \) are its global maximizers, and if it has a minimum then \( x_5 \) is its global minimizer. Whether or not it has a maximum or a minimum, \( x_1 \) and \( x_4 \) are local maximizers, and \( x_5 \) and \( x_4 \) are local minimizers.

15. a. The first-order condition is
\[ g'(Q^*) = c + h^\prime \int_0^Q f(D) dD - p^\prime \int_0^Q f(D) dD = 0. \]

b. \[ g'(Q) = c + h^\prime \int_0^Q f(D) dD - p^\prime \int_0^Q f(D) dD \] and \[ g''(Q) = (h + p) f(Q) \geq 0 \] for all \( Q \), so \( g \) is convex. Thus \( Q^* > 0 \) solves the problem if and only if it satisfies the first-order condition.

c. When \( f(D) = 1/a \) for all \( D \), the first-order condition is

\[
c + (ha)\int_0^Q dD - (pa)\int_0^Q dD = 0, \\
\text{or} \\
c + (ha)Q^* - (pa)(a - Q^*) = 0, \\
\text{or} \\
c + (h+p)Q^*/a - p = 0, \\
\text{or} \\
Q^* = a(p - c)/(h + p).
\]

16. a. First-order condition: \[ g'(\pi(h))\pi'(h) - c'(h) = 0. \]

b. Since \( g \) is increasing and concave and \( \pi \) are concave, \( g(\pi(h)) \) is concave; since \( c \) is convex, \(-c\) is concave. Thus \( g(\pi(h)) - c(h) \) is the sum of two concave functions, and hence is concave. We conclude that an interior solution of the first-order condition is a solution of the problem.

c. The first-order conditions are

\[
p_i f_i'(a_i^*, a_j^*) + p_j g_j'(a_i^*, a_j^*) - 1 = 0 \text{ for all } i = 1,2.
\]

b. Yes: any interior solution of a maximization problem in which the objective function is differentiable satisfies the first-order condition.

c. If \((a_i^*, a_j^*)\) satisfies the first-order conditions and the Hessian of the objective function at \((a_i^*, a_j^*)\) is negative definite then \((a_i^*, a_j^*)\) is a local maximizer. The Hessian is

\[
\begin{pmatrix}
p_i f_{i i}''(a_i^*, a_j^*) + p_j g_{j i}''(a_i^*, a_j^*) & p_i f_{i j}''(a_i^*, a_j^*) + p_j g_{j j}''(a_i^*, a_j^*)
\end{pmatrix}
\]

d. [Note that the condition that the Hessian be negative definite at \((a_i^*, a_j^*)\) is sufficient, not necessary. Note also that this condition is not the same as the objective function being "concave at \((a_i^*, a_j^*)\). If you look at the definition of concavity, you see that every function is concave on a domain consisting of a single point!]

e. If \( f \) and \( g \) are concave then the objective function is concave (since the sum of concave functions is concave), and hence any solution of the first-order
6.1.1 Optimization with an equality constraint: necessary conditions for an optimum for a function of two variables

Motivation

Consider the two-variable problem

\[ \text{max}_{x, y} f(x, y) \text{ subject to } g(x, y) = c. \]

The constraint set (i.e. the set of all pairs \((x, y)\) for which \(g(x, y) = c\)) is a set of points in \((x, y)\) space. Suppose that it is a curve (a one-dimensional set). Assume also that the level curves of \(f\) are one-dimensional, as in the following figure. Assume that \(f\) is increasing, so that \(k' > k\).

\[
\begin{align*}
    f(x, y) &= k \\
    f(x, y) &= k'
\end{align*}
\]

\[
\begin{align*}
    g(x, y) &= c
\end{align*}
\]

Assuming that the functions \(f\) and \(g\) are differentiable, we see from the figure that at a solution \((x^*, y^*)\) of the problem, the constraint curve is tangent to a level curve of \(f\), so that (using the equation for a tangent),

\[
\frac{f'(x^*, y^*)}{g'(x^*, y^*)} = -\frac{g'(x^*, y^*)}{f'(x^*, y^*)}
\]

or

\[
\frac{f'(x^*, y^*)}{g'(x^*, y^*)} = \frac{f'(x^*, y^*)}{g'(x^*, y^*)},
\]

assuming that neither \(g'(x^*, y^*)\) nor \(g'(x^*, y^*)\) is zero.

Now introduce a new variable, \(\lambda\), and set it equal to the common value of the quotients:

\[
\lambda = f'(x^*, y^*)/g'(x^*, y^*) = f'(x^*, y^*)/g'(x^*, y^*)
\]

You might think that introducing a new variable merely complicates the problem, but in fact it is a clever step that allows the
condition for a maximum to be expressed in an appealing way. In addition, the variable turns out to have a very useful interpretation.

Given the definition of \( \lambda \), the condition for \((x^*, y^*)\) to solve the problem may be written as the two equations

\[
\begin{align*}
    f_1'(x^*, y^*) - \lambda g_1'(x^*, y^*) &= 0 \\
    f_2'(x^*, y^*) - \lambda g_2'(x^*, y^*) &= 0.
\end{align*}
\]

At a solution of the problem we need, in addition, \( c = g(x^*, y^*) \) (the constraint is satisfied). Thus the following conditions must be satisfied at a solution \((x^*, y^*)\):

\[
\begin{align*}
    f_1'(x^*, y^*) - \lambda g_1'(x^*, y^*) &= 0 \\
    f_2'(x^*, y^*) - \lambda g_2'(x^*, y^*) &= 0 \\
    c - g(x^*, y^*) &= 0.
\end{align*}
\]

The first two equations can be viewed conveniently as the conditions for the derivatives of the \textbf{Lagrangean}

\[ L(x, y) = f(x, y) - \lambda(g(x, y) - c) \]

with respect to \( x \) and \( y \) to be zero. They are known as the \textbf{first-order conditions} for the problem.

In summary, this argument suggests that if \((x^*, y^*)\) solves the problem

\[
\max_{x, y} f(x, y) \text{ subject to } g(x, y) = c
\]

then, if neither \( g_1'(x^*, y^*) \) nor \( g_2'(x^*, y^*) \) is zero, \((x^*, y^*)\) is a stationary point of the Lagrangean \( L \) (and, of course, the constraint is satisfied).

(This method was developed by Joseph-Louis Lagrange (site 1, site 2, site 3) (1736-1813), born Giuseppe Lodovico Lagrangia in Turin.)

\textbf{Necessary conditions for an optimum}

Precise conditions for an optimum are given in the following result. (Recall that a \textbf{continuously differentiable} function is one whose partial derivatives all exist and are continuous.)

\textbf{Proposition}

Let \( f \) and \( g \) be continuously differentiable functions of two variables defined on the set \( S \), let \( c \) be a number, and suppose that \((x^*, y^*)\) is an interior point of \( S \) that solves the problem

\[
\max_{x, y} f(x, y) \text{ subject to } g(x, y) = c
\]

or the problem

\[
\min_{x, y} f(x, y) \text{ subject to } g(x, y) = c
\]

or is a local maximizer or minimizer of \( f(x, y) \) subject to \( g(x, y) = c \). Suppose also that either \( g_1'(x^*, y^*) \neq 0 \) or \( g_2'(x^*, y^*) \neq 0 \).
Then there is a unique number $\lambda$ such that $(x^*, y^*)$ is a stationary point of the Lagrangean

$$L(x, y) = f(x, y) - \lambda(g(x, y) - c)$$
That is, $(x^*, y^*)$ satisfies the first-order conditions

$$L'_1(x^*, y^*) = f'_1(x^*, y^*) - \lambda g'_1(x^*, y^*) = 0$$
$$L'_2(x^*, y^*) = f'_2(x^*, y^*) - \lambda g'_2(x^*, y^*) = 0.$$  
In addition, $g(x^*, y^*) = c$.

We deduce from this result that the following procedure may be used to solve a maximization problem of the type we are considering.

**Procedure for solving a two-variable maximization problem with an equality constraint**

Let $f$ and $g$ be continuously differentiable functions of two variables defined on a set $S$ and let $c$ be a number. If the problem $\max_{x, y} f(x, y)$ subject to $g(x, y) = c$ has a solution, it may be found as follows.

- Find all the values of $(x, y, \lambda)$ in which (a) $(x, y)$ is an interior point of $S$ and (b) $(x, y, \lambda)$ satisfies the first-order conditions and the constraint (the points $(x, y, \lambda)$ for which $f'_1(x, y) - \lambda g'_1(x, y) = 0$, $f'_2(x, y) - \lambda g'_2(x, y) = 0$, and $g(x, y) = c$).
- Find all the points $(x, y)$ that satisfy $g'_1(x, y) = 0$, $g'_2(x, y) = 0$, and $g(x, y) = c$. (For most problems, there are no such values of $(x, y)$. In particular, if $g$ is linear there are no such values of $(x, y)$.)
- If the set $S$ has any boundary points, find all the points that solve the problem $\max_{x, y} f(x, y)$ subject to the two conditions $g(x, y) = c$ and $(x, y)$ is a boundary point of $S$.
- The points $(x, y)$ you have found at which $f(x, y)$ is largest are the maximizers of $f$.

As before, the variant of this procedure in which the last step involves choosing the points $(x, y)$ at which $f(x, y)$ is smallest may be used to solve the analogous minimization problem.

**Example**

Consider the problem

$$\max_{x, y} xy$$
subject to $x + y = 6$,

where the objective function $xy$ is defined on the set of all 2-vectors, which has no boundary.

The constraint set is not bounded, so the extreme value theorem does not imply that this problem has a solution.
The Lagrangean is
\[ L(x, y) = xy - \lambda (x + y - 6) \]
so the first-order conditions are
\[ L'_1(x, y) = y - \lambda = 0 \]
\[ L'_2(x, y) = x - \lambda = 0 \]
and the constraint is \( x + y - 6 = 0 \).

These equations have a unique solution, \((x, y, \lambda) = (3, 3, 3)\). We have \( g'(x, y) = 1 \neq 0 \) and \( g''(x, y) = 1 \neq 0 \) for all \((x, y)\), so we conclude that if the problem has a solution it is \((x, y) = (3, 3)\).

**Example**

Consider the problem
\[ \max x^2 y \]
subject to \(2x^2 + y^2 = 3\),
where the objective function \(x^2y\) is defined on the set of all 2-vectors, which has no boundary.

The constraint set is compact and the objective function is continuous, so the extreme value theorem implies that the problem has a solution.

The Lagrangean is
\[ L(x, y) = x^2y - \lambda (2x^2 + y^2 - 3) \]
so the first-order conditions are
\[ L'_1(x, y) = 2xy - 4\lambda x = 2x(y - 2\lambda) = 0 \]
\[ L'_2(x, y) = x^2 - 2\lambda y = 0 \]
and the constraint is \(2x^2 + y^2 - 3 = 0\).

To find the solutions of these three equations, first note that from the first equation we have either \(x = 0\) or \(y = 2\lambda\). We can check each possibility in turn.

- \(x = 0\): we have \(y = 3^{1/2}\) and \(\lambda = 0\), or \(y = -3^{1/2}\) and \(\lambda = 0\).
- \(y = 2\lambda\): we have \(x^2 = y^2\) from the second equation, so either \(x = 1\) or \(x = -1\) from the third equation.
  - \(x = 1\): either \(y = 1\) and \(\lambda = 1/2\), or \(y = -1\) and \(\lambda = -1/2\).
  - \(x = -1\): either \(y = 1\) and \(\lambda = 1/2\), or \(y = -1\) and \(\lambda = -1/2\).
In summary, the first-order conditions have six solutions:

1. \((x, y, \lambda) = (0, 3^{1/2}, 0)\), with \(f(x, y) = 0\).
2. \((x, y, \lambda) = (0, -3^{1/2}, 0)\), with \(f(x, y) = 0\).
3. \((x, y, \lambda) = (1, 1, 1/2)\), with \(f(x, y) = 1\).
4. \((x, y, \lambda) = (1, -1, -1/2)\), with \(f(x, y) = -1\).
5. \((x, y, \lambda) = (-1, 1, 1/2)\), with \(f(x, y) = 1\).
6. \((x, y, \lambda) = (-1, -1, -1/2)\), with \(f(x, y) = -1\).

Now, \(g_1'(x, y) = 4x\) and \(g_2'(x, y) = 2y\), so the only value of \((x, y)\) for which \(g_1'(x, y) = 0\) and \(g_2'(x, y) = 0\) is \((x, y) = (0, 0)\). At this point the constraint is not satisfied, so the only possible solutions of the problem are the solutions of the first-order conditions.

We conclude that the problem has two solutions, \((x, y) = (1, 1)\) and \((x, y) = (-1, 1)\).

The next problem is an example of a consumer’s maximization problem, with utility function \(x^a y^b\) and budget constraint \(px + y = m\).

Example

Consider the problem

\[
\max_{x, y} x^a y^b \text{ subject to } px + y = m,
\]

where \(a > 0, b > 0, p > 0,\) and \(m > 0,\) and the objective function \(x^a y^b\) is defined on the set of all points \((x, y)\) with \(x \geq 0\) and \(y \geq 0\).

The Lagrangean is

\[
L(x, y) = x^a y^b - \lambda(px + y - m)
\]

so the first-order conditions are

\[
ax^{a-1}y^b - \lambda p = 0 \\
bx^a y^{b-1} - \lambda = 0
\]

and the constraint is \(px + y = m\). From the first two conditions we have \(ay = pbx\). Substituting into the constraint we obtain

\[
x = \frac{aml((a + b)p)}{ab} \quad \text{and} \quad y = \frac{bml(a + b)}{ab}
\]

so that

\[
\lambda = \frac{[ar^b/(a+b)^{ab-1}][m^{ab-1}/p^r]}.
\]

The value of the objective function at this point is \([am/((a+b)p)]\cdot[bml/(a+b)]\), which is positive.

We have \(g_1'(x, y) = p\) and \(g_2'(x, y) = 1\), so there are no values of \((x, y)\) for which
\[ g'(x, y) = g'(x, y) = 0. \]

The boundary of the set on which the objective function is defined is the set of points \((x, y)\) with \(x = 0\) or \(y = 0\). At every such point the value of the objective function is 0.

We conclude that if the problem has a solution, it is \((x, y) = (am/(a + b)p), bm/(a + b))\).

**Example**

Consider the problem
\[
\max_{x, y} x \text{ subject to } x^2 = 0,
\]
where the objective function is defined on the set of all points \((x, y)\).

The Lagrangean is

\[ L(x, y) = x - \lambda x^2 \]

so the first-order conditions are

\[
\begin{align*}
1 - 2\lambda x &= 0 \\
0 &= 0
\end{align*}
\]

and the constraint is \(x^2 = 0\). From the constraint we have \(x = 0\), which does not satisfy the first first-order condition. Thus the three equations have no solution.

Now, we have \(g'(x, y) = 2x\) and \(g'(x, y) = 0\). Thus we have \(g'(x, y) = g'(x, y) = 0\) for \((x, y) = (0, y)\) for any value of \(y\). At all these points the value of the objective function is the same (namely 0). Hence if the problem has a solution, the set of solutions is the set of pairs \((0, y)\) for all values of \(y\).

(In fact the problem has a solution, though we cannot deduce that it does from the Extreme Value Theorem, because the constraint set, which consists of all pairs \((0, y)\), for any value of \(y\), is not compact.)

### 6.1.1 Exercises on necessary condition for an optimum for a problem with an equality constraint

1. Consider the problem

\[
\max_{x, y} x^2 + y^2 \text{ subject to } x^2 + xy + y^2 = 3,
\]

and the similar problem in which \(\max\) is replaced by \(\min\). Does the extreme value theorem show that these problems have solutions? Use the Lagrangean method to find the solutions.

2. Find the solution of

\[
\min_{x, y} (x-1)^2 + y^2 \text{ subject to } y^2 - 8x = 0,
\]

taking as given that the problem has a solution.
3. What does the Lagrangean procedure say about the solutions of each of the following problems?

\[
\text{max}_x, y - 2x^2 + x \text{ subject to } (x + y)^2 = 0
\]

and

\[
\text{max}_x, y - 2x^2 + x \text{ subject to } x + y = 0
\]

4. Solve the problem

\[
\text{max}_{x, y} \, xy \text{ subject to } x^2 + y^2 = 2c^2,
\]

where \(c > 0\) is a constant. Be sure to lay out all the steps of your argument!

5. Solve the problem

\[
\text{max}_{x, y} \, x + 2y \text{ subject to } 2x^2 + y^2 = 18.
\]

Be sure to lay out carefully the steps in your argument.

### 6.1.1 Solutions to exercises on necessary condition for an optimum for a problem with an equality constraint

1. The objective function is continuous (it is a polynomial). We can write the constraint as \((x + (1/2)y)^2 + (3/4)y^2 = 3\). Each of the terms on the left hand side is nonnegative, so certainly at any point that satisfies the constraint we have \((3/4)y^2 \leq 3\), or \(-2 \leq y \leq 2\). At any point that satisfies the constraint we have also \((x + (1/2)y)^2 \leq 3\), so that \(-\sqrt{3} \leq x + (1/2)y \leq \sqrt{3}\), and hence, given \(-2 \leq y \leq 2\), we have \(-\sqrt{3} - 1 \leq x \leq \sqrt{3} + 1\). Thus the constraint set is bounded. The constraint set is also closed (it is defined by an equality). Thus the extreme value theorem shows that the problems have solutions.

The first-order conditions and the constraint are

\[
\begin{align*}
2x - \lambda(2x + y) &= 0 \\
2y - \lambda(x + 2y) &= 0 \\
x^2 + xy + y^2 &= 3
\end{align*}
\]

The first two equations imply that \(x^2 = y^2\), so that either \(x = y\) or \(x = -y\). Now substitute for \(y\) in the third equation to conclude that the three equations have four solutions: \((1, 1, 2/3), (-1, -1, 2/3), (3^{1/2}, -3^{1/2}, 2), (-3^{1/2}, 3^{1/2}, 2)\). The unique solution of \(g'(x, y) = g(x, y) = 0\) (where \(g\) is the function in the constraint) is \((x, y) = (0, 0)\), which does not satisfy the constraint, so the only candidates for solutions of the problems are the four solutions of the first-order conditions.

The values of the objective function at these four points are 2, 2, 6, and 6. Thus, given that the two problems have solutions, the solutions of the minimization
problem are \((x, y) = (1, 1)\) and \((x, y) = (-1, -1)\), and the solutions of the maximization problem are \((x, y) = (3^{1/2}, -3^{1/2})\) and \((x, y) = (-3^{1/2}, 3^{1/2})\).

2. The first-order conditions and the constraint are

\[
\begin{align*}
2(x - 1) + 8\lambda &= 0 \\
2y - 2\lambda y &= 0 \\
y^2 - 8x &= 0.
\end{align*}
\]

3. The unique solution of these conditions is \((x, y, \lambda) = (0, 0, 1/4)\). The equations \(g_1'(x, y) = g_2'(x, y) = 0\) have no solution. Thus if the problem has a solution, it is \((x, y) = (0, 0)\).

4. First problem: The first-order conditions and the constraint are

\[
\begin{align*}
-4x + 1 - 2\lambda(x + y) &= 0 \\
1 - 2\lambda(x+y) &= 0 \\
(x + y)^2 &= 0
\end{align*}
\]

5. These equations have no solution. Let \(g(x, y) = (x + y)^2\), the constraint function. The equations \(g_1'(x, y) = g_2'(x, y) = 0\) (have \((x, -x)\) as solutions for any value of \(x\), and all such solutions satisfy the constraint. Thus the Lagrangean procedure says nothing about the solution (except that it satisfies the constraint!).

6. Second problem: The first-order conditions and the constraint are

\[
\begin{align*}
-4x + 1 - \lambda &= 0 \\
1 - \lambda &= 0 \\
x + y &= 0
\end{align*}
\]

7. These equations have the unique solution \((x, y, \lambda) = (0, 0, 1)\). Let \(g(x, y) = (x + y)^2\), the constraint function. The equations \(g_1'(x, y) = g_2'(x, y) = 0\) have no solutions. Thus the Lagrangean procedure says that if the problem has a solution (which in fact it does) then that solution is \((x, y) = (0, 0)\).

8. NOTE: The two problems are equivalent (any point that satisfies the constraint in the first problem satisfies the constraint in the second problem, and vice versa), so in fact the solution of the second problem is the unique solution of the first problem.

9. The objective function is continuous, and the constraint set is closed and bounded (it is a circle), so by the extreme value theorem the problem has a solution. Any solution of the problem satisfies the constraint and either the first-order conditions or the condition that \(g_1'(x, y) = g_2'(x, y) = 0\), where \(g\) is the constraint function.

Now, \(g_1'(x, y) = 2x\) and \(g_2'(x, y) = 2y\), so that the only solution of \(g_1'(x, y) = g_2'(x, y) = 0\) is \((x, y) = (0, 0)\). This solution does not satisfy the constraint, so we conclude that any solution of the problem satisfies the first-order conditions. The first-order conditions and the constraint are
\[ \begin{align*}
y - 2\lambda x &= 0 \\
x - 2\lambda y &= 0 \\
x^2 + y^2 &= 2c^2.
\end{align*} \]

10. From the first-order conditions, either \( y = 0 \) or \( \lambda^2 = 1/4 \). If \( y = 0 \) then \( x = 0 \), so that the constraint is not satisfied. Thus we have \( \lambda^2 = 1/4 \), so that either \( \lambda = 1/2 \) or \( \lambda = -1/2 \). Thus the first-order conditions have four solutions: \((c, c)\), \((c, -c)\), \((-c, c)\), and \((-c, -c)\). The value of the objective function at \((c, c)\) and \((-c, -c)\) is \( c^2 \), while its value at \((c, -c)\) and \((-c, c)\) is \(-c^2\). Thus the problem has two solutions: \((c, c)\) and \((-c, -c)\).

11. The constraint set is compact and the objective function is continuous, so the problem has a solution. Suppose that \((x^*, y^*)\) solves the problem. Then we know that \( g(x^*, y^*) = c \) and either there exists \( \lambda \) such that \((x^*, y^*, \lambda)\) satisfies the first-order conditions or \( \nabla g(x^*, y^*) = (0, 0) \). The first-order conditions are

\[ \begin{align*}
1 - 4\lambda x &= 0 \\
2 - 2\lambda y &= 0
\end{align*} \]

12. which imply that \( 4x = y \). Combined with the constraint we deduce that \((x, y) = (1, 4)\) or \((x, y) = (-1, -4)\). We have \( \nabla g(x, y) = (4x, 2y) \), which is \((0, 0)\) only if \((x, y) = (0, 0)\), which violates the constraint. Since \( f(1, 4) = 9 > f(-1, -4) = -9 \), the unique solution of the problem is \((x, y) = (1, 4)\).

### 6.1.2 Optimization with an equality constraint: interpretation of Lagrange multiplier

Consider the problem

\[ \max_{xy} f(x, y) \text{ subject to } g(x, y) = c. \]

Suppose we solve the problem for various values of \( c \). Let the solution be \((x^*(c), y^*(c))\) with a Lagrange multiplier of \( \lambda^*(c) \). Assume that the functions \( x^*, y^*, \) and \( \lambda^* \) are differentiable and that \( g'(x^*(c), y^*(c)) \neq 0 \) or \( g'(x^*(c), y^*(c)) \neq 0 \), so that the first-order conditions are satisfied.

Let \( f^*(c) = f(x^*(c), y^*(c)) \). Differentiate \( f^*(c) \) with respect to \( c \):

\[ f^*(c) = f'(x^*(c), y^*(c))x^*(c) + f'(x^*(c), y^*(c))y^*(c) = \lambda^*(c)[g'(x^*(c), y^*(c))x^*(c) + g'(x^*(c), y^*(c))y^*(c)] \]

(using the first-order conditions). But \( g(x^*(c), y^*(c)) = c \) for all \( c \), so the derivatives of each side of this equality are the same. That is,

\[ g'(x^*(c), y^*(c))x^*(c) + g'(x^*(c), y^*(c))y^*(c) = 1 \text{ for all } c. \]

Hence

\[ f^*(c) = \lambda^*(c). \]
That is,

the value of the Lagrange multiplier at the solution of the problem is equal to the rate of change in the maximal value of the objective function as the constraint is relaxed.

For example, in a utility maximization problem the optimal value of the Lagrange multiplier measures the marginal utility of income: the rate of increase in maximized utility as income is increased.

Example
Consider the problem
\[ \max_{x^c} x^c \text{ subject to } x = c. \]
The solution of this problem is obvious: \( x = c \) (the only point that satisfies the constraint!). The maximized value of the function is thus \( c^c \), so that the derivative of this maximized value with respect to \( c \) is \( 2c \).

Let's check that the value of the Lagrange multiplier at the solution of the problem is equal to \( 2c \). The Lagrangean is

\[ L(x) = x^c - \lambda(x - c), \]
so the first-order condition is

\[ 2x - \lambda = 0. \]
The constraint is \( x = c \), so the pair \((x, \lambda)\) that satisfies the first-order condition and the constraint is \((c, 2c)\). Thus we see that indeed \( \lambda \) is equal to the derivative of the maximized value of the function with respect to \( c \).

Example
A firm uses two inputs to produce one output. Its production function is

\[ f(x, y) = x^a y^b, \]
The price of output is \( p \), and the prices of the inputs are \( w_x \) and \( w_y \). The firm is constrained by a law that says it must use exactly the same number of units of both inputs.

Thus the firm's problem is

\[ \max_{x,y} [px^a y^b - w_x x - w_y y] \text{ subject to } y - x = 0. \]
(The firm is also constrained by the conditions \( x \geq 0 \) and \( y \geq 0 \), but I am ignoring these constraints at the moment.)

The Lagrangean is

\[ L(x,y) = px^a y^b - w_x x - w_y y - \lambda(y - x) \]
so the first-order conditions are
\[ apx^{w-1}y^w - w, + \lambda = 0 \]
\[ bpxy^{w-1} - w, - \lambda = 0 \]

and the constraint is \( y = x \). These equations have a single solution, with

\[ x = y = ((w, + w,)/(p(a + b)))^{1/(a+b-1)} \]

and

\[ \lambda = (bw, - aw,)/(a + b). \]

There is no value of \((x, y)\) for which \( g_1'(x, y) = g_2'(x, y) = 0 \), so if the problem has a solution it is the solution of the first-order condition.

Since \( \lambda \) measures the rate of increase of the maximal value of the objective function as the constraint is relaxed, it follows that if \( \lambda > 0 \) then the firm would like the constraint to be relaxed: its profit would be higher if the constraint were \( y - x = \varepsilon \), for some \( \varepsilon > 0 \).

Suppose that \( bw, > aw, \), so that \( \lambda > 0 \), and the firm would like to use more of input \( y \) than of input \( x \). A government inspector indicates that for a bribe, she is willing to overlook a small violation of the constraint: she is willing to allow the firm to use a small amount more of input \( y \) than it does of input \( x \). Suppose the constraint is relaxed to \( y - x = \varepsilon \). The maximum bribe the firm is willing to offer is the increase in its maximized profit, which is approximately \( \varepsilon \lambda = \varepsilon (bw, - aw,)/(a + b) \). Hence this is the maximum bribe the firm is willing to pay. (If \( w, = w, = 1, a = 1/4, \) and \( b = 1/2, \) for example, the maximum bribe is \( \varepsilon/3 \).)

### 6.1.2 Exercises on interpretation of Lagrange multiplier

1. A firm that uses two inputs to produce output has the production function \( 3x^{1/3}y^{2/3} \), where \( x \) is the amount of input 1 and \( y \) is the amount of input 2. The price of output is 1 and the prices of the inputs are \( w, \) and \( w, \). The firm is constrained by the government to use exactly 1000 units of input 1.
   a. How much of input 2 does it use?
   b. What is the most that it is willing to bribe an inspector to allow it to use another unit of input 1?

2. A firm that uses two inputs to produce output has the production function \( 4x^{1/4}y^{3/4} \). The price of output is 1 and the price of each input is 1. The firm is constrained to use exactly 1000 units of input \( x \).
   a. How much of input \( y \) does it use?
b. What is approximately the maximum amount the firm is willing to pay to be allowed to use \( \varepsilon \) more units of input \( x \), for \( \varepsilon \) small? (Do not try to calculate your answer as a decimal number.)

6.1.2 Solutions to exercises on interpretation of Lagrange multiplier

1. a. Solving the first-order conditions we obtain \( y^* = (10/w_y)^{3/2} \).

b. The maximal bribe is given by the value of the Lagrange multiplier, \( \lambda^* = (1/100) \cdot (10/w_y)^{3/2} - w_x \). (If this is negative, then the firm isn't willing to pay any bribe to increase the amount of input 1 that it uses--it is instead willing to pay a bribe of \( \lambda^* \) to decrease the amount of that input. You may verify that if \( w_x \) and \( w_y \) are such that in the absence of a constraint the firm chooses to use 1000 units of input 1 then \( \lambda^* = 0 \) in the constrained problem.)

2. a. The firm’s profit is \( 4x^{3/4}y^{1/4} - x - y \). Thus its problem is

\[
\max_{x,y} 4x^{3/4}y^{1/4} - x - y \text{ subject to } x = 1000.
\]

The first-order conditions for this problem are

\[
\begin{align*}
x^{3/4}y^{1/4} - 1 - \lambda &= 0 \\
3x^{1/4}y^{3/4} - 1 &= 0
\end{align*}
\]

Thus \((x, y) = (1000, 81000)\). Also we have \( \lambda = (81)^{3/4} - 1 = 26 \).

b. If the firm is allowed to use \( \varepsilon \) more units of input \( x \) then its profit increases by approximately \( \lambda \varepsilon \), or \( 26 \varepsilon \).

6.1.3 Optimization with an equality constraint: sufficient conditions for a local optimum for a function of two variables

Consider the problem

\[
\max_{x,y} f(x, y) \text{ subject to } g(x, y) = c.
\]

Assume that \( g'(x, y) \neq 0 \). By substituting for \( y \) using the constraint, we can reduce the problem to one in a single variable, \( x \). Let \( h \) be implicitly defined by \( g(x, h(x)) = c \). Then the problem is

\[
\max_x f(x, h(x)).
\]

Define \( F(x) = f(x, h(x)) \). Then

\[
F'(x) = f'(x, h(x)) + f'(x, h(x))h'(x).
\]

Let \( x^* \) be a stationary point of \( F \) (i.e. \( F'(x^*) = 0 \)). A sufficient condition for \( x^* \) to be a local maximizer of \( F \) is that \( F''(x^*) < 0 \). We have
\[ F''(x^*) = f_1''(x^*, h(x^*)) + 2f_1''(x^*, h(x^*))h'(x^*) + f_1''(x^*, h(x^*))h''(x^*) + f_2''(x^*, h(x^*))h'(x^*) + f_2''(x^*, h(x^*))h''(x^*). \]

Now, since \( g(x, h(x)) = c \) for all \( x \), we have

\[
g'(x, h(x)) + g'(x, h(x))h'(x) = 0,
\]

so that

\[
h'(x) = \frac{-g'(x, h(x))}{g'(x, h(x))}.
\]

Using this expression we can find \( h''(x^*) \), and substitute it into the expression for \( F''(x^*) \). After some manipulation, we find that

\[
F''(x^*) = \frac{-D(x^*, y^*, \lambda^*)}{(g_i'(x^*, y^*))^2}
\]

where

\[
D(x^*, y^*, \lambda^*) = \begin{vmatrix}
0 & g_i'(x^*, y^*) & g_i'(x^*, y^*) \\
g_i'(x^*, y^*) & f_1''(x^*, y^*) - \lambda g_i''(x^*, y^*) & f_1''(x^*, y^*) - \lambda g_i''(x^*, y^*) \\
g_i'(x^*, y^*) & f_1''(x^*, y^*) - \lambda g_i''(x^*, y^*) & f_1''(x^*, y^*) - \lambda g_i''(x^*, y^*)
\end{vmatrix}
\]

and \( \lambda^* \) is the value of the Lagrange multiplier at the solution (i.e. \( f_i'(x^*, y^*)/g_i'(x^*, y^*) \)).

The matrix of which \( D(x^*, y^*, \lambda^*) \) is the determinant is known as the bordered Hessian of the Lagrangean.

In summary, we have the following result.

**Proposition**

Consider the problems

\[
\max_{x, y} f(x, y) \text{ subject to } g(x, y) = c
\]

and

\[
\min_{x, y} f(x, y) \text{ subject to } g(x, y) = c.
\]

Suppose that \((x^*, y^*)\) and \( \lambda^* \) satisfy the first-order conditions

\[
\begin{align*}
f_i'(x^*, y^*) - \lambda g_i'(x^*, y^*) &= 0 \\
f_j'(x^*, y^*) - \lambda g_j'(x^*, y^*) &= 0
\end{align*}
\]

and the constraint \( g(x^*, y^*) = c \).

- If \( D(x^*, y^*, \lambda^*) > 0 \) then \((x^*, y^*)\) is a local maximizer of \( f \) subject to the
constraint $g(x, y) = c$.
- If $D(x^*, y^*, \lambda^*) < 0$ then $(x^*, y^*)$ is a local minimizer of $f$ subject to the constraint $g(x, y) = c$.

**Example**

Consider the problem

$max_{x, y} xy$ subject to $x + y = 6$.

We argued previously that if this problem has a solution, it is $(3, 3)$. Now we can check if this point is at least a local maximizer.

The Lagrangean is

$$ L(x, y) = xy + \lambda (6 - x - y) $$

so that the determinant of the bordered Hessian of the Lagrangean is

$$ D(x, y) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} $$

The determinant of this matrix is $1 + 1 = 2 > 0$, so the point $(3, 3)$ is indeed a local maximizer.

**Example**

Consider the problem

$max_{x, y} x^2 y$ subject to $2x^2 + y^2 = 3$.

We found previously that there are six solutions of the first-order conditions, namely

1. $(x, y, \lambda) = (0, 3^{1/2}, 0)$, with $f(x, y) = 0$.
2. $(x, y, \lambda) = (0, -3^{1/2}, 0)$, with $f(x, y) = 0$.
3. $(x, y, \lambda) = (1, 1, 1/2)$, with $f(x, y) = 1$.
4. $(x, y, \lambda) = (1, -1, -1/2)$, with $f(x, y) = -1$.
5. $(x, y, \lambda) = (-1, 1, 1/2)$, with $f(x, y) = 1$.
6. $(x, y, \lambda) = (-1, -1, -1/2)$, with $f(x, y) = -1$.

Further, we found that solutions 3 and 5 are global maximizers, while solutions 4 and 6 are global minimizers.

The two remaining solutions of the first-order conditions, $(0, 3^{1/2})$ and $(0, -3^{1/2})$, are neither global maximizers nor global minimizers. Are they local maximizers or local minimizers?
The determinant of the bordered Hessian of the Lagrangean is

\[ D(x, y, \lambda) = \begin{vmatrix} 0 & 4x & 2y \\ 4x & 2y - 4\lambda & 2x \\ 2y & 2x & -2\lambda \end{vmatrix} \]

This determinant is

\[ -4x(-8x\lambda - 4xy) + 2y(8x^2 - 2y(2y - 4\lambda)) = 8[2\lambda(2x^2 + y^2) + y(4x^2 - y^2)] = 8[6\lambda + y(4x^2 - y^2)] \]

(since \(2x^2 + y^2 = 3\) at each solution, from the constraint). The value of the determinant at the two solutions is

- (0, \(3^{1/2}\), 0): \(-8\cdot3^{3/2}\), so (0, \(3^{1/2}\)) is a local minimizer;
- (0, \(-3^{1/2}\), 0): \(8\cdot3^{1/2}\), so (0, \(-3^{1/2}\)) is a local maximizer.

### 1.3 Exercises on sufficient conditions for a local optimum for a problem with an equality constraint

1. For the problem

max \( x^2 + y^2 \) subject to \( x^2 + xy + y^2 = 3 \)

find all the solutions of the first-order conditions and determine, if possible, whether each solution is a local maximizer or a local minimizer.

### 6.1.3 Solutions to exercises on sufficient conditions for a local optimum for a problem with an equality constraint

1. The bordered Hessian is

\[
\begin{pmatrix}
0 & 2x + y & x + 2y \\
2x + y & 2 - 2\lambda & -\lambda \\
x + 2y & -\lambda & 2 - 2\lambda.
\end{pmatrix}
\]

2. The determinant of this matrix is \(-2[5x^2 + 8xy + 5y^2] + 6\lambda[x^2 + xy + y^2]\). Evaluate this at the solutions of the first-order conditions (found in a previous problem):
   - (1, 1, 2/3): determinant is \(-24 < 0\), so (1, 1) is a local minimizer.
o \((-1, -1, 2/3)\): determinant is \(-24 < 0\), so \((-1, -1)\) is a local minimizer.

o \((\sqrt{3}, -\sqrt{3}, 2)\): determinant is \(24 > 0\), so \((\sqrt{3}, -\sqrt{3})\) is a local maximizer.

o \((-\sqrt{3}, \sqrt{3}, 2)\): determinant is \(24 > 0\), so \((-\sqrt{3}, \sqrt{3})\) is a local maximizer.

6.1.4 Optimization with an equality constraint: conditions under which a stationary point is a global optimum

We know that if \((x^*, y^*)\) solves the problem
\[
\max_{x, y} f(x, y) \text{ subject to } g(x, y) = c
\]
and either \(g_1'(x^*, y^*) \neq 0\) or \(g_2'(x^*, y^*) \neq 0\) then there is a number \(\lambda^*\) such that \((x^*, y^*)\) is a stationary point of the Lagrangean \(L(x, y) = f(x, y) - \lambda^*(g(x, y) - c))\), given \(\lambda^*\).

The fact that \((x^*, y^*)\) is a stationary point of the Lagrangean does not mean that \((x^*, y^*)\) maximizes the Lagrangean, given \(\lambda^*\). (The Lagrangean is a function like any other, and we know that a stationary point of an arbitrary function is not necessarily a maximizer of the function. In an exercise you are asked to work through a specific example.)

Suppose, however, that \((x^*, y^*)\) does in fact maximize \(L(x, y)\), given \(\lambda^*\). Then
\[
L(x^*, y^*) \geq L(x, y) \text{ for all } (x, y),
\]
or
\[
f(x^*, y^*) - \lambda^*(g(x^*, y^*) - c) \geq f(x, y) - \lambda^*(g(x, y) - c) \text{ for all } (x, y).
\]
Now, if \((x^*, y^*)\) satisfies the constraint then \(g(x^*, y^*) = c\), so this inequality is equivalent to
\[
f(x^*, y^*) \geq f(x, y) - \lambda^*(g(x, y) - c) \text{ for all } (x, y),
\]
so that
\[
f(x^*, y^*) \geq f(x, y) \text{ for all } (x, y) \text{ with } g(x, y) = c.
\]
That is, \((x^*, y^*)\) solves the constrained maximization problem.

In summary,

if \((x^*, y^*)\) maximizes the Lagrangean, given \(\lambda^*\), and satisfies the constraint, then it solves the problem.

Now, we know that any stationary point of a concave function is a maximizer of the function. Thus if the Lagrangean is concave in \((x, y)\), given \(\lambda^*\), and \((x^*, y^*)\) is a stationary point of the Lagrangean, then \((x^*, y^*)\) maximizes the Lagrangean, given \(\lambda^*\), and hence if it satisfies the constraint then it solves the problem. Precisely, we have the following result.

Proposition

Suppose that \(f\) and \(g\) are continuously differentiable functions defined on an open convex subset \(S\) of two-dimensional space and suppose that there exists a number
\( \lambda^* \) such that \((x^*, y^*)\) is an interior point of \(S\) that is a stationary point of the Lagrangean

\[
L(x, y) = f(x, y) - \lambda^*(g(x, y) - c).
\]

Suppose further that \(g(x^*, y^*) = c\). Then

- if \(L\) is concave---in particular if \(f\) is concave and \(\lambda^* g\) is convex---then \((x^*, y^*)\) solves the problem \(\max_{x,y} f(x, y)\) subject to \(g(x, y) = c\)
- if \(L\) is convex---in particular if \(f\) is convex and \(\lambda^* g\) is concave---then \((x^*, y^*)\) solves the problem \(\min_{x,y} f(x, y)\) subject to \(g(x, y) = c\).

Note that if \(g\) is linear then \(\lambda^* g\) is both convex and concave, regardless of the value of \(\lambda^*\). Thus if \(f\) is concave and \(g\) is linear, every interior point of \(S\) that is a stationary point of the Lagrangean is a solution of the problem.

Example

Consider the problem

\[
\max_{x,y} xy \text{ subject to } px + y = m,
\]

considered previously. We found that there is a value of \(\lambda^*\) such that

\[
(x^*, y^*) = \begin{pmatrix}
am \\
(bm) \\
(a + b)p \\
(a + b)
\end{pmatrix}
\]

is a stationary point of the Lagrangean and satisfies the constraint. Now, if \(a \geq 0, b \geq 0,\) and \(a + b \leq 1\) then the objective function is concave; the constraint is linear, so from the result above, \((x^*, y^*)\) is a solution of the problem.

6.1.4 Exercises on conditions under which a stationary point is a global optimum

1. Consider the problem

\[
\max_{x,y} xy \text{ subject to } x + y = 2.
\]

Show that for \(\lambda = 1\) the Lagrangean has a stationary point at \((x, y) = (1, 1)\). Show also that \((x, y) = (1, 1)\) does not maximize the Lagrangean function \(L(x, y) = xy - 1 \cdot (x + y - 2)\).

2. A firm's production function is \(f(x_i, x_i)\) and the prices of its inputs are \(w_i > 0\) and \(w_i > 0; f\) is concave and increasing \((f'/(x_i, x_i) > 0\) for each \(i\), for all \((x_i, x_i))\). The firm wishes to minimize the cost of producing \(q\) units of output.
   a. Write down the firm's optimization problem.
   b. Write down the first-order conditions for the problem.
c. If \((x_1, x_2)\) satisfies the first-order conditions and the constraint, is it necessarily a solution of the problem? Justify your answer carefully.

6.1.4 Solutions to exercises on conditions under which a stationary point is a global optimum

1. For \(\lambda = 1\) the derivatives of the Lagrangean are zero if and only if

\[
\begin{align*}
y - 1 &= 0 \\
x - 1 &= 0,
\end{align*}
\]

which yield \((x, y) = (1, 1)\).

2. The point \((x, y) = (1, 1)\) does not maximize the Lagrangean with \(\lambda = 1\) because \(L(1, 1) = 1\), whereas \(L(2, 2) = 2\), for example.

4. a. \(\text{min}_{x_1, x_2}(w_1 x_1 + w_2 x_2)\) subject to \(f(x_1, x_2) = q\).

b. \(w_1 - \lambda f'(x) = 0\)

\(w_2 - \lambda f'(x) = 0\)

c. From the first-order conditions we have \(\lambda = w_i / f(x_1, x_2) > 0\). The objective function is convex, and the constraint function is concave, so the Lagrangean is convex. Thus if \((x_1^*, x_2^*)\) satisfies the first-order conditions and the constraint then it is a solution to the problem.

6.2 Optimization with equality constraints: \(n\) variables, \(m\) constraints

The Lagrangean method can easily be generalized to a problem of the form

\[
\text{max } f(x) \text{ subject to } g_j(x) = c_j \text{ for } j = 1, \ldots, m
\]

with \(n\) variables and \(m\) constraints (where \(x = (x_1, \ldots, x_n)\)).

The Lagrangean for this problem is

\[
L(x) = f(x) - \sum_{j=1}^{m} \lambda_j (g_j(x) - c_j).
\]

That is, there is one Lagrange multiplier for each constraint.

As in the case of a problem with two variables and one constraint, the first-order condition is that \(x^*\) be a stationary point of the Lagrangean. The "nondegeneracy" condition in the two variable case---namely that at least one of \(g'_1(x_1, x_2)\) and \(g'_2(x_1, x_2)\) is nonzero---is less straightforward to generalize. The appropriate generalization involves the "rank" of the \(n \times m\) matrix in which the \((i, j)\)th component is the partial derivative of
g, with respect to \( x \) evaluated at \( x^* \). None of the problems require you to know how to apply this condition.

**Proposition (Necessary conditions for an extremum)**

Let \( f \) and \( g_1, \ldots, g_m \) be continuously differentiable functions of \( n \) variables defined on the set \( S \), let \( c_j \) for \( j = 1, \ldots, m \) be numbers, and suppose that \( x^* \) is an interior point of \( S \) that solves the problem

\[
\text{max. } f(x) \text{ subject to } g_j(x) = c_j \text{ for } j = 1,\ldots,m.
\]

or the problem

\[
\text{min. } f(x) \text{ subject to } g_j(x) = c_j \text{ for } j = 1,\ldots,m.
\]

or is a local maximizer or minimizer of \( f(x) \) subject to \( g_j(x) = c_j \) for \( j = 1, \ldots, m \). Suppose also that the rank of the ("Jacobian") matrix in which the \((i,j)\)th component is \( \frac{\partial g_j}{\partial x_i}(x^*) \) is \( m \).

Then there are unique numbers \( \lambda_1, \ldots, \lambda_m \) such that \( x^* \) is a stationary point of the Lagrangean function \( L \) defined by

\[
L(x) = f(x) - \sum_{j=1}^m \lambda_j (g_j(x) - c_j).
\]

That is, \( x^* \) satisfies the first-order conditions

\[
L'(x^*) = f'(x^*) - \sum_{j=1}^m \lambda_j (\frac{\partial g_j}{\partial x_i}(x^*)) = 0 \text{ for } i = 1,\ldots,n.
\]

In addition, \( g_j(x^*) = c_j \) for \( j = 1, \ldots, m \).

As in the case of a problem with two variables and one constraint, the first-order conditions and the constraint are sufficient for a maximum if the Lagrangean is concave, and are sufficient for a minimum if the Lagrangean is convex, as stated precisely in the following result.

**Proposition (Conditions under which necessary conditions for an extremum are sufficient)**

Suppose that \( f \) and \( g_j \) for \( j = 1, \ldots, n \) are continuously differentiable functions defined on an open convex subset \( S \) of \( n \)-dimensional space and let \( x^* \in S \) be an interior stationary point of the Lagrangean

\[
L(x) = f(x) - \sum_{j=1}^m \lambda_j (g_j(x) - c_j).
\]

Suppose further that \( g_j(x^*) = c_j \) for \( j = 1, \ldots, m \). Then

- if \( L \) is concave---in particular if \( f \) is concave and \( \lambda^*_j g_j \) is convex for \( j = 1, \ldots, m \)---then \( x^* \) solves the constrained maximization problem
- if \( L \) is convex---in particular if \( f \) is convex and \( \lambda^*_j g_j \) is concave for \( j = 1, \ldots, m \)---then \( x^* \) solves the constrained minimization problem
Example

Consider the problem
\[ \min_{x, y, z} x^2 + y^2 + z^2 \] subject to \( x + 2y + z = 1 \) and \( 2x - y - 3z = 4 \).

The Lagrangean is
\[ L(x, y, z) = x^2 + y^2 + z^2 - \lambda_1 (x + 2y + z - 1) - \lambda_2 (2x - y - 3z - 4). \]

This function is convex for any values of \( \lambda_1 \) and \( \lambda_2 \), so that any interior stationary point is a solution of the problem. Further, the rank of the Jacobian matrix is 2 (a fact you can take as given), so any solution of the problem is a stationary point. Thus the set of solutions of the problem coincides with the set of stationary points.

The first-order conditions are
\[
\begin{align*}
2x - \lambda_1 - 2\lambda_2 &= 0 \\
2y - 2\lambda_1 + \lambda_2 &= 0 \\
2z - \lambda_1 + 3\lambda_2 &= 0
\end{align*}
\]

and the constraints are
\[
\begin{align*}
x + 2y + z &= 1 \\
2x - y - 3z &= 4
\end{align*}
\]

Solve first two first-order conditions for \( \lambda_1 \) and \( \lambda_2 \) to give
\[
\lambda_1 = \frac{2}{5}x + \frac{4}{5}y,
\lambda_2 = \frac{4}{5}x - \frac{2}{5}y.
\]

Now substitute into last the first-order condition and then use the two constraints to get
\[
x = \frac{16}{15}, \; y = \frac{1}{3}, \; z = -\frac{11}{15},
\text{with } \lambda_1 = \frac{52}{75} \text{ and } \lambda_2 = \frac{54}{75}.
\]

We conclude that \((x, y, z) = (16/15, 1/3, -11/15)\) is the unique solution of the problem.

Economic Interpretation of Lagrange Multipliers

In the case of a problem with two variables and one constraint we saw that the Lagrange multiplier has an interesting economic interpretation. This interpretation generalizes to the case of a problem with \( n \) variables and \( m \) constraints.
Consider the problem

\[
\max f(x) \text{ subject to } g_j(x) = c_j \text{ for } j = 1, \ldots, m,
\]
where \( x = (x_1, \ldots, x_n) \). Let \( x^*(c) \) be the solution of this problem, where \( c \) is the vector \((c_1, \ldots, c_m)\) and let

\[
f^*(c) = f(x^*(c)).
\]
Then we have

\[
f_j^*(c) = \lambda_j(c)
\]
for \( j = 1, \ldots, m \), where \( \lambda_j \) is the value of the Lagrange multiplier on the \( j \)th constraint at the solution of the problem.

That is:

the value of the Lagrange multiplier on the \( j \)th constraint at the solution of the problem is equal to the rate of change in the maximal value of the objective function as the \( j \)th constraint is relaxed.

If the \( j \)th constraint arises because of a limit on the amount of some resource, then we refer to \( \lambda_j(c) \) as the shadow price of the \( j \)th resource.

### 6.2 Exercises on equality-constrained optimization problems with many variables and constraints

1. Solve the problem

\[
\max_{x, y} (x + y) \text{ subject to } x^2 + 2y^2 + z^2 = 1 \text{ and } x + y + z = 1
\]
(assuming, without checking, that the "nondegeneracy condition" is satisfied).

2. Consider the problem

\[
\max f(x) \text{ subject to } p \cdot x = c
\]
where \( x = (x_1, \ldots, x_n) \), \( f \) is concave, \( p = (p_1, \ldots, p_n) \), \( p_i > 0 \) for \( i = 1, \ldots, n \), and \( c \) is a number.
   
   a. Write down the first-order conditions for a solution of this problem.
   
   b. What is the relation (if any) between an interior solution of the first-order conditions and a solution of the problem?
   
   c. Solve the problem in the case \( n = 2 \) and \( f(x_1, x_2) = -x_1^2 - 2x_2^2 \) (which is concave).

3. Consider the problem

\[
\max W(t) - c(t) \text{ subject to } g(t) = k,
\]
where \( t = (t_1, \ldots, t_n) \), \( W \) is differentiable and concave, and \( c \) is differentiable and convex, \( g \) is differentiable and increasing (for every \( i \), \( g'(t) > 0 \) for all \( t \)), and \( k \) is a constant.
a. Write down the first-order conditions for a solution of this problem.

b. What is the relation (if any) between a solution of the first-order conditions and a solution of the problem?

c. Solve the problem for \( n = 2 \), \( W(t_1, t_2) = -(t_1 - 1)^2 - t_2^2 \), \( c(t_1, t_2) = t_1 + t_2 \), \( g(t_1, t_2) = t_1 + t_2 \), and an arbitrary value of \( k \). Give an interpretation to the value of the Lagrange multiplier you find.

6.2 Solutions to exercises on equality-constrained optimization problems with many variables and constraints

1. By the extreme value theorem the problem has a solution. (The objective function is continuous because it is a polynomial (in fact, it is a linear function), and the constraint set is compact because it is the intersection of a plane and the surface of a sphere.)

The first-order conditions are

\[
\begin{align*}
1 - 2\lambda_i x - \lambda_i & = 0 \\
1 - 4\lambda_i y - \lambda_i & = 0 \\
-2\lambda_i z - \lambda_i & = 0 
\end{align*}
\]

and the constraints are \( x^2 + 2y^2 + z^2 = 1 \) and \( x + y + z = 1 \). To find values of \( x, y, z, \lambda_i \), and \( \lambda \), that solve these equations, we can first use the third first-order condition to eliminate \( \lambda_i \), and then use the second constraint to eliminate \( z \). Then we obtain

\[
\begin{align*}
1 - \lambda_i [4x + 2y - 2] & = 0 \\
1 - \lambda_i [2x + 6y - 2] & = 0 \\
2x^2 + 3y^2 - 2y + 2xy - 2x & = 0 
\end{align*}
\]

The first of these two equations yields \( x = 2y \), so that the second equation is \( 3y(5y - 2) = 0 \), so that either \( y = 0 \) or \( y = 2/5 \).

Thus there are two solutions of the first-order conditions and the constraints, namely \((0, 0, 1)\) with \( \lambda_i = -1/2 \) and \( \lambda_i = 1 \), and \((4/5, 2/5, -1/5)\) with \( \lambda_i = 5/18 \) and \( \lambda_i = 1/9 \).

Now, the objective function \( x + y \) is concave and each constraint is convex, so that \( \lambda c_i g_j \) is convex for each \( j \) for the second solution. Thus the second solution is the solution of the problem. [Alternatively, you can check that the value of the function is higher at the second solution.]

2.

a. \( f'(x) - \lambda p_i = 0 \) for some \( \lambda \), for \( i = 1, ..., n \).

b. Since \( p_i > 0 \) for all \( i \), there is no point at which the gradient of the constraint function is zero. Thus if \( x^* \) solves the problem then there exists
\( \lambda \) such that \((x^*, \lambda)\) solve the first-order conditions. Since \( f \) is concave and \( g \) is linear, if \((x^*, \lambda)\) solves the first-order conditions, then \( x^* \) solves the problem. Thus \( x^* \) solves the problem if and only if there exists \( \lambda \) such that \((x^*, \lambda)\) solves the first-order conditions.

c. The first-order conditions are

\begin{align*}
-2x_1 - \lambda p_1 &= 0 \\
-4x_2 - \lambda p_2 &= 0
\end{align*}

d. and the constraint is \( p_1 x_1 + p_2 x_2 = c \). The unique solution of these three equations is

e. \((x_1, x_2) = (2p_1c/(2p_1^2 + p_2^2), p_2c/(2p_1^2 + p_2^2))\).

3.

a. \( W'(t) - c'(t) - \lambda g'(t) = 0 \) for some \( \lambda \) for \( i = 1, \ldots, n \).

b. If \( t^* \) solves the problem then \( g(t^*) = k \) and there exists \( \lambda \) such that the first-order conditions are satisfied. (Note that there is no value of \( t \) for which \( \nabla g(t) = (0, \ldots, 0) \).) If \((t^*, \lambda)\) solves the first-order conditions, \( g(t^*) = k \), and \( \lambda g \) is convex, then \( t^* \) is a solution of the problem (given the concavity of \( W(t) - c(t) \) and the convexity of \( g \)).

c. The first-order conditions are

\begin{align*}
-2(t_1 - 1) - 1 - \lambda &= 0 \\
-2t_2 - 1 - \lambda &= 0.
\end{align*}

d. We deduce that \( t_1 - 1 = t_2 \), so that from the constraint \( t_1 + t_2 = k \) we have \( t_1 = (1/2)(k + 1) \) and \( t_2 = (1/2)(k - 1) \). We have also \( \lambda = -k \). Because \( g \) is linear, \( \lambda g \) is convex, so the solution of the problem is \((t_1, t_2) = ((1/2)(k + 1), (1/2)(k - 1))\). The Lagrange multiplier is equal to the rate of change of the maximal value of the objective function, \( W(t^*) - c(t^*) \), with respect to \( k \).

### 6.3 The envelope theorem

In economic theory we are often interested in how the maximal value of a function depends on some parameters.

Consider, for example, a firm that can produce output with a single input using the production function \( f \). The standard theory is that the firm chooses the amount \( x \) of the input to maximize its profit \( pf(x) - wx \), where \( p \) is the price of output and \( w \) is the price of the input. Denote by \( x^*(w, p) \) the optimal amount of the input when the prices are \( w \) and \( p \). An economically interesting question is: how does the firm's maximal profit \( pf(x^*(w, p)) - wx^*(w, p) \) depend upon \( p \)?

We have already answered this question in an earlier example. To do so, we used the chain rule to differentiate \( pf(x^*(w, p)) - wx^*(w, p) \) with respect to \( p \), yielding
\[f(x^*(w, p)) + x^*'(w, p)[p f'(x^*(w, p)) - w],\]
and then used the fact that \(x^*(w, p)\) satisfies the first-order condition \(p f'(x^*(w, p)) - w = 0\) to conclude that the derivative is simply \(f(x^*(w, p))\).

That is, the fact that the value of the variable satisfies the first-order condition allows us to dramatically simplify the expression for the derivative of the firm's maximal profit. In this section I describe results that generalize this observation to an arbitrary maximization problem.

**Unconstrained problems**

Consider the unconstrained maximization problem
\[
\max_x f(x, r),
\]
where \(x\) is an \(n\)-vector and \(r\) is a \(k\)-vector of parameters. Assume that for any vector \(r\) the problem has a unique solution; denote this solution \(x^*(r)\). Denote the maximum value of \(f\), for any given value of \(r\), by \(f^*(r)\):
\[
f^*(r) = f(x^*(r), r).
\]
We call \(f^*\) the **value function**.

**Example**

Let \(n = 1, k = 2\), and \(f(x, r) = x_1 - r_1 x\), where \(0 < r_1 < 1\). This function is concave (look at its second derivative), and any solution satisfies the first-order condition \(r_1 x_1^{r_1 - 1} - r_2 = 0\). Thus \(x^*(r) = (r_1/r_2)^{1/(1-r_1)}\) is the solution of the problem, so that the value function of \(f\) is
\[
f^*(r) = (x^*(r))_1 - r_2 x^*(r) = (r_1/r_2)^{1/(1-r_1)} - r_2 (r_1/r_2)^{1/(1-r_1)}.
\]

We wish to find the derivatives of \(f^*\) with respect to each parameter \(r_h\) for \(h = 1, \ldots, k\). First, because \(x^*(r)\) is a solution of the problem when the parameter vector is \(r\), it satisfies the **first-order conditions**:

\[f_i'(x^*(r), r) = 0 \text{ for } i = 1, \ldots, n.
\]

Now, using the **chain rule** we have
\[
f^*_{r_h}'(r) = \sum_{i=1}^n f_i'(x^*(r), r) \cdot (\partial x^*_i/\partial r_h)(r) + f_{n+1}'(x^*(r), r).
\]

The first term corresponds to the change in \(f^*\) caused by the change in the solution of the problem that occurs when \(r_h\) changes; the second term corresponds to the direct effect of a change in \(r_h\) on the value of \(f\).

Given the first-order conditions, this expression simplifies to
\[
f^*_{r_h}'(r) = f_{n+1}'(x^*(r), r) \text{ for } h = 1, \ldots, k.
\]

Note that the derivative on the right-hand side is the **partial** derivative of \(f\) with respect to \(r_h\) (the \(n + h\)th variable in the vector \((x, r)\)), **holding \(x\) fixed at \(x^*(r)\**). This result is stated precisely as follows.

**Proposition (Envelope theorem for an unconstrained maximization problem)**

Let \(f\) be a continuously differentiable function of \(n + k\) variables. Define the
function $f^*$ of $k$ variables by

$$f^*(r) = \max_{x} f(x, r),$$

where $x$ is an $n$-vector and $r$ is a $k$-vector. If the solution of the maximization problem is a continuously differentiable function of $r$ then

$$f^*'(r) = f'_{w}(x^*(r), r)$$

for $h = 1, ..., k$.

This result says that the change in the maximal value of the function as a parameter changes is the change caused by the direct impact of the parameter on the function, holding the value of $x$ fixed at its optimal value; the indirect effect, resulting from the change in the optimal value of $x$ caused by a change in the parameter, is zero.

The next two examples illustrate how the result simplifies the calculation of the derivatives of the value function.

Example

Consider the problem studied at the start of this section, in which a firm can produce output, with price $p$, using a single input, with price $w$, according to the production function $f$. The firm's profit when it uses the amount $x$ of the input is

$$\pi(x, (w, p)) = pf(x) - wx,$$

and its maximal profit is

$$f^*(w, p) = pf(x^*(w, p)) - wx^*(w, p),$$

where $x^*(w, p)$ is the optimal amount of the input at the prices $(w, p)$. This function $f^*$ is known as the firm's profit function. By the envelope theorem, the derivative of this function with respect to $p$ is the partial derivative of $\pi$ with respect to $p$ evaluated at $x = x^*(w, p)$, namely

$$f'(x^*(w, p)).$$

In particular, the derivative is positive: if the price of output increases, then the firm's maximal profit increases.

Also by the envelope theorem the derivative of the firm's maximal profit with respect to $w$ is

$$-x^*(p, w).$$

(This result is known as Hotelling's Lemma.) In particular, this derivative is negative: if the price of the input increases, then the firm's maximal profit decreases.

A consequence of Hotelling's Lemma is that we can easily find the firm's input demand function $x^*$ if we know the firm's profit function, even if we do not know the firm's production function: we have $x^*(p, w) = -\pi^*_{,p}(p, w)$ for all $(p, w)$, so we may obtain the input demand function by simply differentiating the profit function.

Example

Consider the earlier example, in which $f(x, r) = x^*_1 - r, x$, where $0 < r, < 1$. We found that the solution of the problem

$$\max_{x} f(x, r)$$


is given by
\[ x^*(r) = (r/r_2)^{1/(1-r_1)}. \]
Thus by the envelope theorem, the derivative of the maximal value of \( f \) with respect to \( r \), is the derivative of \( f \) with respect to \( r \), evaluated at \( x^*(r) \), namely
\[ (x^*(r))' \ln x^*(r), \]
or
\[ (r/r_2)^{1/(1-r_1)} \ln (r/r_2)^{1/(1-r_1)}. \]
(If you have forgotten how to differentiate \( x^r \) with respect to \( r \), (not with respect to \( x! \)), remind yourself of the rules.)

If you approach this problem directly, by calculating the value function explicitly and then differentiating it, rather than using the envelope theorem, you are faced with the task of differentiating
\[ (x^*(r))' - r x^*(r) = (r/r_2)^{1/(1-r_1)} - r(r/r_2)^{1/(1-r_1)} \]
with respect to \( r \), which is much more difficult than the task of differentiating the function \( f \) partially with respect to \( r \).

Why is the result called the envelope theorem? The American Heritage Dictionary (3ed) gives one meaning of "envelope" to be "A curve or surface that is tangent to every one of a family of curves or surfaces". In the following figure, each black curve is the graph of \( f \) as a function of \( r \) for a fixed values of \( x \). (Only a few values of \( x \) are considered; one can construct as many as one wishes.) Each of these graphs shows how \( f \) changes as \( r \) changes, for a given value of \( x \). To find the solution of the maximization problem for any given value of \( r \), we find the highest function for that value of \( r \). For example, for \( r = r' \), the highest function is the one colored blue. The graph of the value function \( f^* \) is the locus of these highest points; it is the envelope of the graphs for each given value of \( x \). From the figure, the envelope theorem is apparent: the slope of the envelope at any given value of \( r \) is the slope of the graph of \( f(x^*(r), r) \). (For example, the slope of the envelope at \( r' \) is the slope of the blue curve at \( r' \).)

**Constrained problems**
We may apply the same arguments to maximization problems with constraints. Consider the problem
\[ \text{max. } f(x, r) \text{ subject to } g(x, r) = 0 \text{ for } j = 1, ..., m, \]
where \( x \) is an \( n \)-vector and \( r \) is a \( k \)-vector of parameters. Assume that for any value of \( r \) the problem has a single solution, and denote this solution \( x^*(r) \). As before, denote the maximum value of \( f \), for any given value of \( r \), by \( f^*(r) \), and call \( f^* \) the value function: 
\[
 f^*(r) = f(x^*(r), r).
\]

Define the Lagrangean \( L \) by
\[
 L(x, r) = f(x, r) - \sum_{i=1}^{m} \lambda_i g_i(x, r),
\]
where \( \lambda_i \), for \( j = 1, ..., m \) is the Lagrange multiplier associated with the solution of the maximization problem.

Now, if the constraints satisfy a "nondegeneracy" condition on the rank of the Jacobian matrix then by an earlier result there are unique numbers \( \lambda_1, ..., \lambda_m \) such that the solution \( x^*(r) \) satisfies the first-order conditions
\[
 f'(x^*(r), r) - \sum_{i=1}^{m} \lambda_i (\partial g_i / \partial x_j)(x^*(r), r) = 0 \quad \text{for} \quad i = 1, ..., n.
\]

We want to calculate the derivatives \( f^*_h(r) \) for \( h = 1, ..., k \) of the function \( f^* \). Using the chain rule, we have
\[
 f^*_h(r) = \sum_{i=1}^{m} f'(x^*(r), r) \cdot (\partial x_i / \partial r_j)(r) + f'_{\lambda_i}(x^*(r), r) 
\]
\[
 = \sum_{i=1}^{m} \left[ \sum_{i=1}^{m} \lambda_i (\partial g_i / \partial x_j)(x^*(r), r) \cdot (\partial x_i / \partial r_j)(r) + f'_{\lambda_i}(x^*(r), r) \right] + f'_{\lambda_j}(x^*(r), r),
\]

where the second line uses the first-order condition. If we reverse the sums we obtain
\[
 f^*_h(r) = \sum_{i=1}^{m} \lambda_i [\sum_{i=1}^{m} (\partial g_i / \partial x_j)(x^*(r), r) \cdot (\partial x_j / \partial r_i)(r)] + f'_{\lambda_j}(x^*(r), r).
\]

But we have \( g(x^*(r), r) = 0 \) for all \( r \), so differentiating with respect to \( r \) we get
\[
\sum_{i=1}^{m} (\partial g_i / \partial x_j)(x^*(r), r) \cdot (\partial x_j / \partial r_i)(r) + (\partial g_i / \partial r_i)(x^*(r), r) = 0.
\]

Hence
\[
 f^*_h(r) = -\sum_{i=1}^{m} \lambda_i (\partial g_i / \partial r_j)(x^*(r), r) + f'_{\lambda_j}(x^*(r), r)
\]
\[
 = \mathbf{L}'_{\lambda_j}(x^*(r), r).
\]

This result is stated precisely as follows.

**Proposition (Envelope theorem for constrained maximization problems)**

Let \( f \) and \( g \), for \( j = 1, ..., m \) be continuously differentiable functions of \( n + k \) variables. Define the function \( f^* \) of \( k \) variables by
\[
 f^*(r) = \max_{x} f(x, r) \quad \text{subject to} \quad g(x, r) = 0 \quad \text{for} \quad j = 1, ..., m,
\]
where \( x \) is an \( n \)-vector and \( r \) is a \( k \)-vector. Suppose that the solution of the maximization problem and the associated Lagrange multipliers \( \lambda_i, ..., \lambda_m \), are continuously differentiable functions of \( r \) and the rank of the ("Jacobian") matrix
\[
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_m
\end{bmatrix}
\]

in which the \( (i, j) \)th component is \( (\partial g_i / \partial x_j)(x^*) \) is \( m \). Then
\[
 f^*_h(r) = \mathbf{L}'_{\lambda_j}(x^*(r), r) \quad \text{for} \quad h = 1, ..., k,
\]
where the function \( L \) is defined by
\[
 L(x, r) = f(x, r) - \sum_{i=1}^{m} \lambda_i g_i(x, r) \quad \text{for every} \quad (x, r).
\]

**Example**

Consider a utility maximization problem:
\[
 \max_{x} u(x) \quad \text{subject to} \quad p \cdot x = w.
\]

where \( x \) is a vector (a bundle of goods), \( p \) is the price vector, and \( w \) is the consumer's wealth (a real number). Denote the solution of the problem by \( x^*(p, w) \), and denote the value function by \( v \), so that
\[ v(p, w) = u(x^*(p, w)) \text{ for every } (p, w). \]

The function \( v \) is known as the **indirect utility function**.

By the envelope theorem we have

\[ v'(p, w) = -\lambda(p, w)x^*(p, w) \]

(since \( u \) does not depend independently on \( p \) or \( w \)) and

\[ \left( \frac{\partial v}{\partial w} \right)(p, w) = \lambda(p, w). \]

Thus

\[ \frac{(\partial v/\partial p)(p, w)}{(\partial v/\partial w)(p, w)} = -x^*(p, w). \]

That is, if you know the indirect utility function then you can

### 6.3 Exercises on the envelope theorem

1. Define the function \( f \) by \( f(x, r) = x^{1/2} - rx \), where \( x \geq 0 \). On a graph with \( r \) on the horizontal axis, sketch the function for several values of \( x \). (E.g. sketch the functions \((1/2)^{1/2} - r/2\), \(1^{1/2} - r\), and \(2^{1/2} - 2r\).) Sketch, in addition, the value function \( f^* \), where \( f^*(r) \) is the maximal value of \( f(x, r) \) for each given value of \( r \).

2. A firm's output depends upon the amount \( x \) of an input and a parameter \( a \), according to the function \( f(x, a) \). This function \( f \) is increasing in \( a \). The price of the firm's output is \( p > 0 \) and the price of the input is \( w > 0 \). Determine the sign of the derivative of the firm's maximal profit with respect to \( a \).

3. The output of a good is \( xy \), where \( x \) and \( y \) are the amounts of two inputs and \( a > 1 \) is a parameter. A government-controlled firm is directed to maximize output subject to meeting the constraint \( 2x + y = 12 \).
   - Solve the firm's problem.
   - Use the envelope theorem to find how the maximal output changes as the parameter \( a \) va

### 6.3 Solutions to exercises on the envelope theorem

1. The first-order condition for a solution of the problem of maximizing \( f \) is \((1/2)x^{1/2} = r\), so that the solution of the problem for a given value of \( r \) is \( x^*(r) = 1/(4r^2) \). Thus \( f^*(r) = 1/4r \). The following figure shows the graph of this function,
in red, together with the graphs of $f(x, r)$ as a function of $r$ for various values of $x$.

2. Denote the firm's maximal profit by $\pi(a, w, p)$. By the envelope theorem we have $\pi'(a, w, p) = pf'_2(x, a)$. Thus since $f$ is increasing in $a$ and $p > 0$, the derivative of the firm's maximal profit with respect to $a$ is positive.

3. The first-order conditions are

$$ax^{-y} - 2\lambda = 0$$
$$x - \lambda = 0$$
$$2x + y = 12$$

4. Solving these we find two solutions: $(x^*, y^*, \lambda^*) = (6a/(1 + a), 12/(1 + a), (6a/(1 + a))^y)$ and $(x', y', \lambda') = (0, 12, 0)$.

5. Since $\nabla g(x^*, y^*) \neq (0, 0)$ the first-order conditions are necessary. That is, if $x^*$ solves the problem then it must solve the first-order conditions. By looking at the geometry of the problem, you can convince yourself that indeed the problem has a solution. Since the output from the first solution exceeds the output from the second solution, the solution to the problem is $(x^*, y^*)$.

6. Let $M(a) = (x^*(a))y^*(a)$. By the envelope theorem we have

$$M'(a) = (\partial L/\partial a)(x^*(a), y^*(a), a)$$
$$= (x^*(a))y^*(a)\ln x^*(a)$$
$$= [12(6a)/\ln (6a/(1 + a))]/(1 + a)^{y^*}$$

7.1 Optimization with inequality constraints: the Kuhn-Tucker conditions
Many models in economics are naturally formulated as optimization problems with *inequality* constraints.

Consider, for example, a consumer’s choice problem. There is no reason to insist that a consumer spend all her wealth, so that her optimization problem should be formulated with inequality constraints:

\[
\max \, u(x) \quad \text{subject to} \quad p \cdot x \leq w \quad \text{and} \quad x \geq 0.
\]

Depending on the character of the function \( u \) and the values of \( p \) and \( w \), we may have \( p \cdot x < w \) or \( p \cdot x = w \) at a solution of this problem.

One approach to solving this problem starts by determining which of these two conditions holds at a solution. In more complex problems, with more than one constraint, this approach does not work well. Consider, for example, a consumer who faces two constraints (perhaps money and time). Three examples are shown in the following figure, which should convince you that we cannot deduce from simple properties of \( u \) alone which of the constraints, if any, are satisfied with equality at a solution.

The problem of a consumer facing two constraints

We consider a problem of the form

\[
\max \, f(x) \quad \text{subject to} \quad g_j(x) \leq c_j \quad \text{for} \quad j = 1, \ldots, m,
\]

where \( f \) and \( g_j \), for \( j = 1, \ldots, m \) are functions of \( n \) variables, \( x = (x_1, \ldots, x_n) \), and \( c_j \) for \( j = 1, \ldots, m \) are constants.

All the problems we have studied so far may be put into this form.

**Equality constraints**

We introduce two inequality constraints for every equality constraint. For example, the problem

\[
\max \, f(x) \quad \text{subject to} \quad h(x) = 0
\]

may be written as
\[
\max_x f(x) \text{ subject to } h(x) \leq 0 \text{ and } -h(x) \leq 0.
\]

Nonnegativity constraints
For a problem with a constraint \( x_i \geq 0 \) we let \( g_i(x) = -x_i \) and \( c_i = 0 \) for some \( j \).

Minimization problems
For a minimization problem we multiply the objective function by \(-1\):
\[
\min_x h(x) \text{ subject to } g_j(x) \leq c_j \text{ for } j = 1, \ldots, m
\]
is the same as
\[
\max_x f(x) \text{ subject to } g_j(x) \leq c_j \text{ for } j = 1, \ldots, m,
\]
where \( f(x) = -h(x) \).

To start thinking about how to solve the general problem, first consider a case with a single constraint (\( m = 1 \)). We can write such a problem as
\[
\max_x f(x) \text{ subject to } g(x) \leq c.
\]

There are two possibilities for the solution of this problem. In the following figures, the black closed curves are contours of \( f \); values of the function increase in the direction shown by the blue arrows. The downward-sloping red line is the set of points \( x \) satisfying \( g(x) = c \); the set of points \( x \) satisfying \( g(x) \leq c \) lie below and to the left of the line, and those satisfying \( g(x) \geq c \) lie above and to the right of the line.

In each panel the solution of the problem is the point \( x^* \). In the left panel the constraint binds at the solution: a change in \( c \) changes the solution. In the right panel, the constraint is slack at the solution: small changes in \( c \) have no effect on the solution.

As before, define the Lagrangean function \( L \) by
\[
L(x) = f(x) - \lambda (g(x) - c).
\]
Then from our previous analysis of problems with equality constraints and with no constraints,
• if \( g(x^*) = c \) (as in the left-hand panel) and the constraint satisfies a regularity condition, then \( L_i'(x^*) = 0 \) for all \( i \)
• if \( g(x^*) < c \) (as in the right-hand panel), then \( f_i'(x^*) = 0 \) for all \( i \).

Now, I claim that in the first case (that is, if \( g(x^*) = c \)) we have \( \lambda \geq 0 \). Suppose, to the contrary, that \( \lambda < 0 \). Then we know that a small decrease in \( c \) raises the maximal value of \( f \). That is, moving \( x^* \) inside the constraint raises the value of \( f \), contradicting the fact that \( x^* \) is the solution of the problem.

In the second case, the value of \( \lambda \) does not enter the conditions, so we can choose any value for it. Given the interpretation of \( \lambda \), setting \( \lambda = 0 \) makes sense. Under this assumption we have \( f_i'(x) = L_i'(x) \) for all \( x \), so that \( L_i'(x^*) = 0 \) for all \( i \).

Thus in both cases we have \( L_i'(x^*) = 0 \) for all \( i, \lambda \geq 0 \), and \( g(x^*) \leq c \). In the first case we have \( g(x^*) = c \) and in the second case \( \lambda = 0 \).

We may combine the two cases by writing the conditions as

\[
L_i'(x^*) = 0 \text{ for } j = 1, \ldots, n \\
\lambda \geq 0, \; g(x^*) \leq c, \; \text{and either } \lambda = 0 \text{ or } g(x^*) - c = 0.
\]

Now, the product of two numbers is zero if and only if at least one of them is zero, so we can alternatively write these conditions as

\[
L_i'(x^*) = 0 \text{ for } j = 1, \ldots, n \\
\lambda \geq 0, \; g(x^*) \leq c, \; \text{and } \lambda[g(x^*) - c] = 0.
\]

The argument I have given suggests that if \( x^* \) solves the problem and the constraint satisfies a regularity condition, then \( x^* \) must satisfy these conditions.

Note that the conditions do not rule out the possibility that both \( \lambda = 0 \) and \( g(x^*) = c \).

The inequalities \( \lambda \geq 0 \) and \( g(x^*) \leq c \) are called complementary slackness conditions; at most one of these conditions is slack (i.e. not an equality).

For a problem with many constraints, then as before we introduce one multiplier for each constraint and obtain the Kuhn-Tucker conditions, defined as follows.

**Definition**

The **Kuhn-Tucker conditions** for the problem

\[
\max f(x) \text{ subject to } g_j(x) \leq c_j \text{ for } j = 1, \ldots, m
\]

are

\[
L_i'(x) = 0 \text{ for } i = 1, \ldots, n
\]
These conditions are named in honor of Harold W. Kuhn, an emeritus member of the Princeton Math Department, and Albert W. Tucker, who first formulated and studied the conditions. In the following sections I discuss results that specify the precise relationship between the solutions of the Kuhn-Tucker conditions and the solutions of the problem. The following example illustrates the form of the conditions in a specific case.

Example
Consider the problem
\[
\max_{x_1, x_2} \left[ -(x_1 - 4)^2 - (x_2 - 4)^2 \right] \text{ subject to } x_1 + x_2 \leq 4 \text{ and } x_1 + 3x_2 \leq 9,
\]
illustrated in the following figure.

We have
\[
L(x_1, x_2) = -(x_1 - 4)^2 - (x_2 - 4)^2 - \lambda_1 (x_1 + x_2 - 4) - \lambda_2 (x_1 + 3x_2 - 9).
\]

The Kuhn-Tucker conditions are
\[
\begin{align*}
-2(x_1 - 4) - \lambda_1 - \lambda_2 &= 0 \\
-2(x_2 - 4) - \lambda_1 - 3\lambda_2 &= 0 \\
x_1 + x_2 &\leq 4, \lambda_1 \geq 0, \text{ and } \lambda_1 (x_1 + x_2 - 4) = 0 \\
x_1 + 3x_2 &\leq 9, \lambda_2 \geq 0, \text{ and } \lambda_2 (x_1 + 3x_2 - 9) = 0.
\end{align*}
\]

7.2 Optimization with inequality constraints: the necessity of the Kuhn-Tucker conditions
We have seen that a solution $x^*$ of an optimization problem with equality constraints is a stationary point of the Lagrangean if the constraints satisfy a regularity condition ($\nabla g(x^*) \neq 0$ in the case of a single constraint $g(x) = c$). In an optimization problem with inequality constraints a related regularity condition guarantees that a solution satisfies the Kuhn-Tucker conditions. The weakest forms of this regularity condition are difficult to verify. The next result gives three alternative strong forms that are much easier to verify.

**Proposition**

Let $f$ and $g_j$ for $j = 1, ..., m$ be continuously differentiable functions of many variables and let $c_j$ for $j = 1, ..., m$ be constants. Suppose that $x^*$ solves the problem

$$\max_x f(x) \text{ subject to } g_j(x) \leq c_j \text{ for } j = 1, ..., m.$$ 

Suppose that

- either each $g_j$ is concave
- or each $g_j$ is convex and there is some $x$ such that $g_j(x) < c_j$ for $j = 1, ..., m$
- or each $g_j$ is quasiconvex, $\nabla g_j(x^*) \neq (0, ..., 0)$ for all $j$, and there is some $x$ such that $g_j(x) < c_j$ for $j = 1, ..., m$.

Then there exists a unique vector $\lambda = (\lambda_1, ..., \lambda_m)$ such that $(x^*, \lambda)$ satisfies the Kuhn-Tucker conditions.

Recall that a linear function is concave, so the conditions in the result are satisfied if each constraint function is linear.

Note that the last part of the second and third conditions is very weak: it requires only that some point strictly satisfy all the constraints.

One way in which the conditions in the result may be weakened is sometimes useful: the conditions on the constraint functions need to be satisfied only by the binding constraints--those for which $g_j(x^*) = c_j$.

The next example shows that some conditions are needed. The problem in the example has a solution, but the Kuhn-Tucker conditions have no solution.
Example
Consider the problem
\[
\max_x x \text{ subject to } y - (1 - x)^3 \leq 0 \text{ and } y \geq 0.
\]

The constraint does not satisfy any of the conditions in the Proposition.

The solution is clearly \((1, 0)\).

The Lagrangean is
\[
L(x) = x - \lambda(y - (1 - x)^3) + \lambda y.
\]

The Kuhn-Tucker conditions are
\[
1 - 3\lambda(1 - x)^2 = 0
\]
\[-\lambda + \lambda_s = 0
\]
\[y - (1 - x)^3 \leq 0, \lambda_s \geq 0, \text{ and } \lambda_s[y - (1 - x)^3] = 0
\]
\[-y \leq 0, \lambda_s \geq 0, \text{ and } \lambda_s[-y] = 0.
\]

These conditions have no solution. From the last condition, either \(\lambda_s = 0\) or \(y = 0\). If \(\lambda_s = 0\) then \(\lambda_s = 0\) from the second condition, so that no value of \(x\) is compatible with the first condition. If \(y = 0\) then from the third condition either \(\lambda_s = 0\) or \(x = 1\), both of which

7.3 Optimization with inequality constraints: the sufficiency of the Kuhn-Tucker conditions

We saw previously that for both an unconstrained maximization problem and a maximization problem with an equality constraint the first-order conditions are sufficient for a global optimum when the objective and constraint functions satisfy appropriate concavity/convexity conditions. The same is true for an optimization problem with inequality constraints. Precisely, we have the following result.

Proposition
Let \(f\) and \(g_j\) for \(j = 1, ..., m\) be continuously differentiable functions of many variables and let \(c_j\) for \(j = 1, ..., m\) be constants. Consider the problem
\[
\max_x f(x) \text{ subject to } g_j(x) \leq c_j \text{ for } j = 1, ..., m.
\]

Suppose that
\[
\begin{itemize}
  \item \(f\) is concave
  \item and \(g_j\) is quasiconvex for \(j = 1, ..., m\).
\end{itemize}

If there exists \(\lambda = (\lambda_1, ..., \lambda_m)\) such that \((x^*, \lambda)\) satisfies the Kuhn-Tucker
conditions then \( x^* \) solves the problem.

This result together with the result giving conditions under which the Kuhn-Tucker conditions are necessary yields the following useful corollary.

**Corollary**

The Kuhn-Tucker conditions are both necessary and sufficient if the objective function is concave and

- either each constraint is linear
- or each constraint function is convex and some vector of the variables satisfies all constraints strictly.

The condition that the objective function be concave is a bit too strong to be useful in some economic applications. Specifically, the assumption we would like to impose on a consumer’s utility function is that it be quasiconcave. The next result is useful in this case.

**Proposition**

Let \( f \) and \( g_j \) for \( j = 1, \ldots, m \) be continuously differentiable functions of many variables and let \( c_j \) for \( j = 1, \ldots, m \) be constants. Consider the problem

\[
\max f(x) \text{ subject to } g_j(x) \leq c_j \text{ for } j = 1, \ldots, m.
\]

Suppose that

- \( f \) is twice differentiable and quasiconcave
- and \( g_j \) is quasiconvex for \( j = 1, \ldots, m \).

If there exists \( \lambda = (\lambda_1, \ldots, \lambda_m) \) and a value of \( x^* \) such that \((x^*, \lambda)\) satisfies the Kuhn-Tucker conditions and \( f'(x^*) \neq 0 \) for \( i = 1, \ldots, n \) then \( x^* \) solves the problem.

The constraints in a standard consumer’s optimization problem are linear, so the following implication of this result and the earlier result giving conditions under which the Kuhn-Tucker conditions are necessary is useful.

**Corollary**

Suppose that the objective function is twice differentiable and quasiconcave and every constraint is linear. If \( x^* \) solves the problem then there exists a unique vector \( \lambda \) such that \((x^*, \lambda)\) satisfies the Kuhn-Tucker conditions, and if \((x^*, \lambda)\) satisfies the Kuhn-Tucker conditions and \( f'(x^*) \neq 0 \) for \( i = 1, \ldots, n \) then \( x^* \) solves the problem.

If you have a minimization problem, remember that you can transform it to a maximization problem by multiplying the objective function by \(-1\). Thus for a
minimization problem the condition on the objective function in the first result above is that it be convex, and the condition in the second result is that it be quasiconvex.

The next two very simple examples illustrate how to use the Kuhn-Tucker conditions.

Example

Consider the problem
\[\max \{-(x - 2)^2\} \text{ subject to } x \geq 1,\]
illustrated in the following figure.

Written in the standard format, this problem is
\[\max \{-(x - 2)^2\} \text{ subject to } 1 - x \leq 0.\]

The objective function is concave and the constraint is linear. Thus the Kuhn-Tucker conditions are both necessary and sufficient: the set of solutions of the problem is the same as the set of solutions of the Kuhn-Tucker conditions.

The Kuhn-Tucker conditions are
\[-2(x - 2) + \lambda = 0\]
\[x - 1 \geq 0, \lambda \geq 0, \text{ and } \lambda(1 - x) = 0.\]

From the last condition we have either \(\lambda = 0\) or \(x = 1\).

- \(x = 1\): \(2 + \lambda = 0\), or \(\lambda = -2\), which violates \(\lambda \geq 0\).
- \(\lambda = 0\): \(-2(x - 2) = 0\); the only solution is \(x = 2\).

Thus the Kuhn-Tucker conditions have a unique solution, \((x, \lambda) = (2, 0)\). Hence the problem has a unique solution \(x = 2\).

Example

Consider the problem
\[\max \{-(x - 2)^2\} \text{ subject to } x \geq 3,\]
illustrated in the following figure.
Written in the standard format, this problem is

$$\max \{-(x - 2)^2\} \text{ subject to } 3 - x \leq 0.$$  

As in the previous example, the objective function is concave and the constraint function is linear, so that the set of solutions of the problem is the set of solutions of the Kuhn-Tucker conditions.

The Kuhn-Tucker conditions are

$$-2(x-2) + \lambda = 0$$
$$x-3 \geq 0, \lambda \geq 0, \text{ and } \lambda(3-x) = 0.$$  

From the last conditions we have either $\lambda = 0$ or $x = 3$.

- $x = 3$: $-2 + \lambda = 0$, or $\lambda = 2$.
- $\lambda = 0$: $-2(x - 2) = 0$; since $x \geq 3$ this has no solution compatible with the other conditions.

Thus the Kuhn-Tucker conditions have a single solution, $(x, \lambda) = (3, 2)$. Hence the problem has a unique solution, $x = 3$.

These two examples illustrate a procedure for finding solutions of the Kuhn-Tucker conditions that is useful in many problems. First, look at the complementary slackness conditions, which imply that either a Lagrange multiplier is zero or a constraint is binding. Then follow through the implications of each case, using the other equations. In the two examples, this procedure is very easy to follow. The following examples are more complicated.

**Example**

Consider the problem

$$\max_{x_1, x_2} \{-(x_1 - 4)^2 - (x_2 - 4)^2\} \text{ subject to } x_1 + x_2 \leq 4 \text{ and } x_1 + 3x_2 \leq 9.$$  

The objective function is concave and the constraints are both linear, so the
We previously found that the Kuhn-Tucker conditions for this problem are

\[-2(x_1 - 4) - \lambda_1 = 0\]
\[-2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0\]
\[x_1 + x_2 \leq 4, \lambda_1 \geq 0, \text{ and } \lambda_1(x_1 + x_2 - 4) = 0\]
\[x_1 + 3x_2 \leq 9, \lambda_2 \geq 0, \text{ and } \lambda_2(x_1 + 3x_2 - 9) = 0.\]

What are the solutions of these conditions? Start by looking at the two conditions
\[\lambda_1(x_1 + x_2 - 4) = 0 \text{ and } \lambda_2(x_1 + 3x_2 - 9) = 0.\] These two conditions yield the following four cases.

- \[x_1 + x_2 = 4 \text{ and } x_1 + 3x_2 = 9: \text{ In this case we have } x_1 = 3/2 \text{ and } x_2 = 5/2. \text{ Then the first two equations are}\]
  \[
  5 - \lambda_1 = 0 \\
  3 - \lambda_1 - 3\lambda_2 = 0
  
  \]
  - which imply that \(\lambda_1 = 6\) and \(\lambda_2 = -1\), which violates the condition \(\lambda_2 \geq 0\).
  - \[x_1 + x_2 = 4 \text{ and } x_1 + 3x_2 < 9, \text{ so that } \lambda_2 = 0: \text{ Then first two equations imply } x_1 = x_2 = 2 \text{ and } \lambda_1 = 4. \text{ All the conditions are satisfied, so } (x_1, x_2, \lambda_1, \lambda_2) = (2, 2, 4, 0) \text{ is a solution.}\]
  - \[x_1 + x_2 < 4 \text{ and } x_1 + 3x_2 = 9, \text{ so that } \lambda_1 = 0: \text{ Then the first two equations imply } x_1 = 12/5 \text{ and } x_2 = 11/5, \text{ violating } x_1 + x_2 < 4.\]
  - \[x_1 + x_2 < 4 \text{ and } x_1 + 3x_2 < 9, \text{ so that } \lambda_1 = \lambda_2 = 0: \text{ Then first two equations imply } x_1 = x_2 = 4, \text{ violating } x_1 + x_2 < 4.\]

So \((x_1, x_2, \lambda_1, \lambda_2) = (2, 2, 4, 0)\) is the single solution of the Kuhn-Tucker conditions. Hence the unique solution of problem is \((x_1, x_2) = (2, 2)\).

The next example involves a problem of the form

\[\max_{x \in \mathbb{R}_+} u(x) \text{ subject to } p \cdot x \leq w, x \geq 0,\]

where \(u\) is quasiconcave, \(p\) is a vector, and \(w\) is a scalar. A standard consumer's maximization problem in economic theory takes this form; the technique used in the example may be used also in problems with other specifications of the function \(u\).

**Example**

Consider the problem

\[\max_{x, y} xy \text{ subject to } x + y \leq 6, x \geq 0, \text{ and } y \geq 0.\]

The objective function is twice-differentiable and quasiconcave and the constraint functions are linear, so the Kuhn-Tucker conditions are necessary and if \(((x^*, y^*), \lambda^*)\) satisfies these conditions and no partial derivative of the objective
function at \((x^*, y^*)\) is zero then \((x^*, y^*)\) solves the problem. Solutions of the Kuhn-Tucker conditions at which all derivatives of the objective function are zero may or may not be solutions of the problem---we need to check the values of the objective function at these solutions.

(Alternatively we can argue as follows. The constraint functions are linear, so the Kuhn-Tucker conditions are necessary. Further, the objective function is continuous and the constraint set is compact, so by the extreme value theorem the problem has a solution. Thus the solutions of the problem are the solutions of the first-order conditions that yield the highest values for the function.)

The Lagrangean is

\[
L(x, y) = xy - \lambda_1(x + y - 6) + \lambda_2x + \lambda_3y.
\]

The Kuhn-Tucker conditions are

\[
\begin{align*}
y &- \lambda_1 + \lambda_2 = 0 \\
x &- \lambda_1 + \lambda_3 = 0 \\
\lambda_1 &\geq 0, x + y \leq 6, \lambda_1(x + y - 6) = 0 \\
\lambda_2 &\geq 0, x \geq 0, \lambda_2x = 0 \\
\lambda_3 &\geq 0, y \geq 0, \lambda_3y = 0.
\end{align*}
\]

- If \(x > 0\) and \(y > 0\) then \(\lambda_3 = \lambda_1 = 0\), so that \(\lambda_2 = x = y\) from the first two conditions. Hence \(x = y = \lambda = 3\) from the third condition. These values satisfy all the conditions.
- If \(x = 0\) and \(y > 0\) then \(\lambda_3 = 0\) from the last condition and hence \(\lambda_1 = x = 0\) from the second condition. But now from the first condition \(\lambda_2 = -y < 0\), contradicting \(\lambda_2 \geq 0\).
- If \(x > 0\) and \(y = 0\) then \(\lambda_3 = 0\), and a symmetric argument yields a contradiction.
- If \(x = y = 0\) then \(\lambda_1 = 0\) form the third set of conditions, so that \(\lambda_2 = \lambda_1\) from the first and second conditions. These values satisfy all the conditions.

We conclude that there are two solutions of the Kuhn-Tucker conditions, \((x, y, \lambda_1, \lambda_2, \lambda_3) = (3, 3, 3, 0, 0)\) and \((0, 0, 0, 0, 0)\). The value of the objective function at \((3, 3)\) is greater than the value of the objective function at \((0, 0)\), so the solution of the problem is \((3, 3)\).
Many of the optimization problems in economic theory have nonnegativity constraints on the variables. For example, a consumer chooses a bundle \( x \) of goods to maximize her utility \( u(x) \) subject to her budget constraint \( p \cdot x \leq w \) and the condition \( x \geq 0 \). The general form of such a problem is

\[
\text{max } f(x) \text{ subject to } g_j(x) \leq c_j \text{ for } j = 1, ..., m \text{ and } x_i \geq 0 \text{ for } i = 1, ..., n.
\]

This problem is a special case of the general maximization problem with inequality constraints, studied previously: the nonnegativity constraint on each variable is simply an additional inequality constraint. Specifically, if we define the function \( g_{m+i} \) for \( i = 1, ..., n \) by \( g_{m+i}(x) = -x_i \) and let \( c_{m+i} = 0 \) for \( i = 1, ..., n \), then we may write the problem as

\[
\text{max } f(x) \text{ subject to } g_j(x) \leq c_j \text{ for } j = 1, ..., m+n
\]

and solve it using the Kuhn-Tucker conditions

\[
L_i'(x) = 0 \text{ for } i = 1, ..., n
\]

\[
\lambda_i \geq 0, \ g(x) \leq c, \text{ and } \lambda_i[g_i(x) - c_i] = 0 \text{ for } j = 1, ..., m+n,
\]

where \( L(x) = f(x) - \sum_i \lambda_i(g_i(x) - c_i) \).

Approaching the problem in this way involves working with \( n + m \) Lagrange multipliers, which can be difficult if \( n \) is large. It turns out that the simple form of the inequality constraints associated with the nonnegativity conditions allows us to simplify the calculations as follows.

First, we form the modified Lagrangean

\[
M(x) = f(x) - \sum_i \lambda_i(g_i(x) - c_i).
\]

Note that this Lagrangean does not include the nonnegativity constraints explicitly. Then we work with the Kuhn-Tucker conditions for the modified Lagrangean:

\[
M'(x) \leq 0, \ x \geq 0, \text{ and } x \cdot M'(x) = 0 \text{ for } i = 1, ..., n
\]

\[
g_i(x) \leq c_i, \lambda_i \geq 0, \text{ and } \lambda_i[g_i(x) - c_i] = 0 \text{ for } j = 1, ..., m.
\]

In an exercise, you are asked to show that if \((x, (\lambda_1, ..., \lambda_m))\) satisfies the original Kuhn-Tucker conditions then \((x, (\lambda_1, ..., \lambda_m))\) satisfies the conditions for the modified Lagrangean, and if \((x, (\lambda_1, ..., \lambda_m))\) satisfies the conditions for the modified Lagrangean then we can find numbers \(\lambda_{m+1}, ..., \lambda_m\) such that \((x, (\lambda_1, ..., \lambda_m))\) satisfies the original set of conditions.

This result means that in any problem for which the original Kuhn-Tucker conditions may be used, we may alternatively use the conditions for the modified Lagrangean. For most problems in which the variables are constrained to be nonnegative, the Kuhn-Tucker conditions for the modified Lagrangean are easier to work with than the conditions for the original Lagrangean.
Example
Consider the problem
\[ \max_{x, y} xy \text{ subject to } x + y \leq 6, \ x \geq 0, \text{ and } y \geq 0 \]

studied in the previous section. The modified Lagrangean is
\[ M(x, y) = xy - \lambda(x + y - 6) \]

and the Kuhn-Tucker conditions for this Lagrangean are
\[
\begin{align*}
y - \lambda & \leq 0, \ x \geq 0, \text{ and } x(y - \lambda) = 0 \\
x - \lambda & \leq 0, \ y \geq 0, \text{ and } y(x - \lambda) = 0 \\
\lambda & \geq 0, \ x + y \leq 6, \ \lambda(x + y - 6) = 0.
\end{align*}
\]

We can find a solution of these conditions as follows.

- If \( x > 0 \) then from the first set of conditions we have \( y = \lambda \). If \( y = 0 \) in this case then \( \lambda = 0 \), so that the second set of conditions implies \( x \leq 0 \), contradicting \( x > 0 \). Hence \( y > 0 \), and thus \( x = \lambda \), so that \( x = y = \lambda = 3 \).
- If \( x = 0 \) then if \( y > 0 \) we have \( \lambda = 0 \) from the second set of conditions, so that the first condition contradicts \( y > 0 \). Thus \( y = 0 \) and hence \( \lambda = 0 \) from the third set of conditions.

We conclude (as before) that there are two solutions of the Kuhn-Tucker conditions, in this case \((x, y, \lambda) = (3, 3, 3)\) and \((0, 0, 0)\). Since the value of the objective function at \((3, 3)\) is greater than the value of the objective function at \((0, 0)\), the solution of the problem is \((3, 3)\).

### 7.5 Optimization: summary of conditions under which first-order conditions are necessary and sufficient

**Unconstrained maximization problems**

\( x^* \) solves max, \( f(x) \)

\( f'(x^*) = 0 \) for \( i = 1, \ldots, n \)

if \( f \) is concave
Equality-constrained maximization problems with one constraint

If \( \nabla g(x^*) \neq (0,...,0) \)

\( x^* \) solves \( \max \ f(x) \) subject to \( g(x) = c \)

there exists \( \lambda \) such that \( L'(x^*) = 0 \) for \( i = 1, ..., n \) and \( g(x^*) = c \)

if \( f \) is concave and \( \lambda g \) is convex

where \( L(x) = f(x) - \lambda (g(x) - c) \).

Inequality-constrained maximization problems

If \( g_j \) is concave for \( j = 1, ..., m \)

or

\( g_j \) is convex for \( j = 1, ..., m \) and there exists \( x \) such that \( g_j(x) < c \) for \( j = 1, ..., m \)

or

\( g_j \) is quasiconvex for \( j = 1, ..., m \), \( \nabla g_j(x^*) \neq (0,...,0) \) for \( j = 1, ..., m \), and there exists \( x \) such that \( g_j(x) < c \) for \( j = 1, ..., m \)

there exists \( (\lambda_1,...,\lambda_m) \) such that \( L'(x^*) = 0 \) for \( i = 1, ..., n \) and \( \lambda_i \geq 0 \), \( g_j(x^*) \leq c_j \), and \( \lambda_j (g_j(x^*) - c_j) = 0 \) for \( j = 1, ..., m \)

if \( g_j \) is quasiconvex for \( j = 1, ..., m \) and either \( f \) is concave

or \( f \) is quasiconcave and twice differentiable and \( \nabla f(x^*) \neq (0,...,0) \)

where \( L(x) = f(x) - \sum_{j=1}^{m} \lambda_j (g_j(x) - c_j) \).

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7.4 Exercises on optimization problems with nonnegativity constraints

1. Solve the problem

\[ \text{max. } xy \text{ subject to } 2x + y \leq 2, \ x \geq 0, \ \text{and} \ y \geq 0. \]

[You may use without proof the fact that \( xy \) is quasiconcave for \( x \geq 0 \) and \( y \geq 0 \).]

2. Consider the problem

\[ \text{max. } f(x) \text{ subject to } g_j(x) \leq c_j \text{ for } j = 1, \ldots, m \text{ and } x_i \geq 0 \text{ for } i = 1, \ldots, n. \]

a. Write down the Kuhn-Tucker conditions for this problem when it is written in the form

\[ \text{max. } f(x) \text{ subject to } g_j(x) \leq c_j \text{ for } j = 1, \ldots, m+n \]

where \( g_m(x) = -x_i \) for \( i = 1, \ldots, n \). (Write the derivative of the Lagrangean explicitly in terms of the derivatives of \( f \) and \( g \) for \( j = 1, \ldots, m \), using the notation \( \frac{\partial g_j}{\partial x_i}(x) \) for the derivative of \( g_j \) with respect to \( x_i \) at \( x \). Denote the Lagrange multiplier associated with the constraint \( g_j(x) \leq c_j \) by \( \lambda_j \) for \( j = 1, \ldots, m \) and the multiplier associated with the constraint \( g_m(x) \leq 0 \) by \( \lambda_m \) for \( i = 1, \ldots, n \).)

b. Write down the Kuhn-Tucker conditions tailored to problems with nonnegativity constraints.

c. Show that if \((x, (\lambda_1, \ldots, \lambda_m))\) satisfies the conditions in (a) then \((x, (\lambda_1, \ldots, \lambda_m))\) satisfies the conditions in (b), and if \((x, (\lambda_1, \ldots, \lambda_m))\) satisfies the conditions in (b) then there exist numbers \( \lambda_m, \ldots, \lambda_m \) such that \((x, (\lambda_1, \ldots, \lambda_m))\) satisfies the conditions in (a).

3. Consider the following problem.

\[ \text{max. } -(x_1-c_1) - (x_2-c_2) \text{ subject to } (x_1 + 1)^2 + x_2^2 \leq 4, \ x_1 \geq 0, \ \text{and} \ x_2 \geq 0, \]

where \( c_1 \) and \( c_2 \) are constants.

a. Are the Kuhn-Tucker conditions necessary for a solution of this problem?

b. Are the Kuhn-Tucker conditions sufficient for a solution of this problem?

c. If possible, use the Kuhn-Tucker conditions to find the solution(s) of the problem for \( c_1 = -1 \) and \( c_2 = 3 \).

d. If possible, use the Kuhn-Tucker conditions to find the solution(s) of the problem for \( c_1 = c_2 = 0 \).

e. If possible, use the Kuhn-Tucker conditions to find the solution(s) of the problem for \( c_1 = 2 \) and \( c_2 = 0 \).
4. Let \( u \) be a quasiconcave, increasing function of \( n \) variables. (Increasing means \( u'(x) > 0 \) for all \( i \).) Let \( w > 0 \) be a number and let \( p = (p_1, ..., p_i) \) be a vector with \( p_i > 0 \) for all \( i \). Consider the problem

\[
\max u(x) \text{ subject to } \sum_{i=1}^n p_i x_i \leq w \text{ and } x \geq 0.
\]

a. Write down the Kuhn-Tucker conditions for this problem.

b. Suppose that \( x^* \) solves the problem. Is there necessarily a value of \( \lambda^* \) such that \( (x^*, \lambda^*) \) satisfies the Kuhn-Tucker conditions?

c. Suppose that \( (x^*, \lambda^*) \) satisfies the Kuhn-Tucker conditions. Does \( x^* \) necessarily solve the problem?

d. Suppose that \( x^* \) solves the problem. What can you say about the relationship between \( p_i/p_j \) and the value of \( u'(x^*)/u'(x^*) \) when

   i. \( x_i^* > 0 \) and \( x_j^* > 0 \)?  
   ii. \( x_i^* = 0 \) and \( x_j^* > 0 \)?  
   iii. \( x_i^* = 0 \) and \( x_j^* = 0 \)?

5. Consider the problem

\[
\max x + 3y \text{ subject to } (x + 1)^2 + (y + 1)^2 \leq 5, \ x \geq 0, \text{ and } \ y \geq 0.
\]

a. Suppose that \( (x^*, y^*) \) solves this problem. Is there necessarily a value of \( \lambda^* \) such that \( (x^*, y^*, \lambda^*) \) satisfies the Kuhn-Tucker conditions?

b. Now suppose that \( (x^*, y^*, \lambda^*) \) satisfies the Kuhn-Tucker conditions. Does \( (x^*, y^*) \) necessarily solve the problem?

c. Given the information in your answers to (a) and (b), use the Kuhn-Tucker method to solve the problem. (You may use a graph to get some idea of what the solution might be; but use the Kuhn-Tucker conditions to find the solution exactly and justify it.)

6. A firm produces in each of \( n \) periods. In any period \( i \) its output is \( x_i \) and the price of output is \( p_i \) (which is fixed, but varies between periods). If it chooses the capacity \( k \), then its output in each period is at most \( k \). The cost of maintaining the capacity \( k \) is \( D(k) \), where \( D \) is convex, and the cost of producing the vector of outputs \( x = (x_1, ..., x_n) \) is \( C(x) \), where \( C \) is convex. Thus the firm’s problem is

\[
\max \sum_{i=1}^n (p_i x_i - C(x_1, ..., x_n) - D(k)) \text{ subject to } 0 \leq x_i \leq k \text{ for } i = 1, ..., n.
\]

(Note that the firm chooses both the vector \( x \) and the number \( k \), and that one constraint is \( k \geq 0 \).)

a. Write down the Kuhn-Tucker conditions (either variety) for this problem.

b. What is the relation between the solutions of the Kuhn-Tucker conditions and the solutions of the problem? (You may use the result in a previous problem.)

c. Consider the case in which \( C(x_1, x_2) = x_1^2 + x_2^2 \), \( D(k) = k^2 \) (both of which are convex), \( p_1 = 1 \), and \( p_2 = 3 \). Is there a solution in which \( x_1 = x_2 = k > 0 \)?

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Is there a solution in which \( 0 < x_1 < k = x_2 \)?

7. Solve the problem

\[
\max_{x,y} (x^{1/2} + y) \text{ subject to } px + y \leq I, \quad x \geq 0, \quad \text{ and } y \geq 0
\]

where \( p > 0 \) and \( I > 0 \) are parameters. [You may use without proof the fact that the function \( x^{1/2} + y \) is quasiconcave.]

8. Consider the problem

\[
\max_{x_1,x_2} f(x_1, x_2) \text{ subject to } p_1 x_1 + p_2 x_2 \leq c, \quad x_1 \geq 0, \quad \text{ and } x_2 \geq 0,
\]

where the function \( f \) is differentiable and quasiconcave, \( p_i > 0 \) for \( i = 1, 2 \), and \( c > 0 \).

a. Write down the Kuhn-Tucker conditions (either variety) for \((x^*_1, x^*_2)\) to solve this problem.

b. If \((x^*_1, x^*_2)\) solves the problem does it necessarily satisfy the Kuhn-Tucker conditions?

c. If \((x^*_1, x^*_2)\) satisfies the Kuhn-Tucker conditions and the constraint does it necessarily solve the problem?

d. Let \((x^*_1(p_1, p_2, c), x^*_2(p_1, p_2, c))\) be the solution of the problem, as a function of the parameters; assume that at this solution the values of \( x^*_i \) are both positive and the constraint is satisfied with equality. Let the value of the Lagrange multiplier be \( \lambda^*(p_1, p_2, c) \). Define the function \( f^*(p_1, p_2, c) = f(x^*_1(p_1, p_2, c), x^*_2(p_1, p_2, c)) \). Use the envelope theorem to find \( F'(p_1, p_2, c) \) in terms of the parameters and functions of these parameters.

e. For \( f(x_1, x_2) = x_1 x_2^2 \), which is quasiconcave (you do not need to show this), find the solution of the problem.

**solutions: 7.4 Solutions to exercises on optimization problems with nonnegativity constraints**

1. The constraints are concave, so the KT conditions are necessary. The objective function is quasiconcave, and the constraint is quasiconvex. So if \( \nabla f(x^*, y^*) \neq (0, \ldots, 0) \) and \((x^*, y^*, \lambda^*)\) satisfies the KT conditions, then \((x^*, y^*)\) is a solution of the problem.

Let \( M(x, y) = x^2 y^2 - \lambda(2x + y - 2) \). The KT conditions are

\[
2xy^2 - 2\lambda \leq 0, \quad x \geq 0, \quad \text{and} \quad x[2xy^2 - 2\lambda] = 0 \\
2x^2y - \lambda \leq 0, \quad y \geq 0, \quad \text{and} \quad y[2x^2y - \lambda] = 0 \\
\lambda \geq 0, \quad 2x + y \leq 2, \quad \text{and} \quad \lambda(2x + y - 2) = 0
\]

Consider each possibility for the solutions of the first two sets of conditions:
The Kuhn-Tucker conditions are

\[
f_i'(x) - \sum_{j=1}^{m} \lambda_j (\partial g_j / \partial x_i)(x) \leq 0, \quad x \geq 0, \quad \text{for } i = 1, \ldots, n
\]

\[
\lambda_j \geq 0, \quad g_j(x) \leq c_j, \quad \lambda_j [g_j(x) - c_j] = 0, \quad \text{for } j = 1, \ldots, m
\]

\[
\lambda_{m+i} \geq 0, \quad x_i \geq 0, \quad \lambda_{m+i} x_i = 0, \quad \text{for } i = 1, \ldots, n.
\]

The function \( g_1 \) is convex; the remaining two constraints are also convex (and concave). Finally, there exists a point that satisfies all the constraints.
strictly (e.g. \(1/2, 1/2\)). Thus the Kuhn-Tucker conditions are necessary for a solution to the problem.

- Since \(f\) is concave and each \(g_j\) is convex, the Kuhn-Tucker conditions are sufficient for a solution of the problem.

- The Lagrangean is

\[
M(x_1, x_2) = -(x_1 - c_1)^2 - (x_2 - c_2)^2 - \lambda[(x_1 + 1)^2 + x_2^2 - 4].
\]

The Kuhn-Tucker conditions are

\[-2(x_1 + 1) - 2\lambda(x_1 + 1) \leq 0, x_1 \geq 0, \text{ and } x_1[-2(x_1 + 1) - 2\lambda(x_1 + 1)] = 0 \]

\[-2(x_2 - 3) - 2\lambda x_2 \leq 0, x_2 \geq 0, \text{ and } x_2[-2(x_2 - 3) - 2\lambda x_2] = 0 \]

\[(x_1 + 1)^2 + x_2^2 \leq 4, \lambda \geq 0, \text{ and } \lambda[4 - (x_1 + 1)^2 - x_2^2] = 0 \]

The unique solution of these conditions is \(x_1 = 0, x_2 = \sqrt{3}, \lambda = \sqrt{3} - 1\). Thus a solution of the problem (in fact the only solution, as one can see from a diagram) is \((x_1, x_2) = (0, \sqrt{3})\).

- In this case the unique solution of the Kuhn-Tucker conditions has \((x_1, x_2) = (0, 0)\), so this is a solution of the problem. (\(\lambda\) is zero in this case.)

- In this case the unique solution of the Kuhn-Tucker conditions has \((x_1, x_2) = (1, 0)\), so this is a solution of the problem. (\(\lambda\) is positive in this case.)

\[-1.\]

- The Kuhn-Tucker conditions are

\[u'(x^*) - \lambda^* p_i \leq 0, x_i^* \geq 0, \text{ and } x_i(u'(x^*) - \lambda^* p_i) = 0, i = 1, \ldots, n\]

\[\sum p_i x_i^* \leq w \leq 0, \lambda^* \geq 0, \text{ and } \lambda^* (w - \sum p_i x_i^*) = 0\]

- The constraint function is linear, hence concave. Thus if \(x^*\) solves the problem then there must be a \(\lambda^*\) such that \((x^*, \lambda^*)\) solves the Kuhn-Tucker conditions.

- The objective function is quasiconcave and increasing (so that there is no point at which \(\nabla u(x) = (0, \ldots, 0)\)) and the constraint function is linear, hence quasiconvex. Thus if \((x^*, \lambda^*)\) satisfies the Kuhn-Tucker conditions then \(x^*\) solves the problem.

- If \(x_i^* > 0\) and \(x_j^* > 0\) then from the Kuhn-Tucker conditions we have

\[u'(x^*)/u'(x^*) = p/p_i.\]

- If \(x_i^* = 0\) and \(x_j^* > 0\) then from the Kuhn-Tucker conditions we have \(u'(x^*) \leq \lambda^* p_i\) and \(u'(x^*) = \lambda^* p_i\), so

\[u'(x^*)/u'(x^*) \leq p/p_i.\]

- If \(x_i^* = 0\) and \(x_j^* = 0\) then from the Kuhn-Tucker conditions we have \(u'(x^*) \leq \lambda^* p_i\) and \(u'(x^*) \leq \lambda^* p_i\), so we can say nothing about the relationship of the ratio of the partials of \(u\) and \(p/p_i\).
-1.

- The constraint function is convex (its Hessian is
\[
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\]
and there exists \((x, y) \geq (0, 0)\) such that \(g(x, y) < c\), so the Kuhn-Tucker conditions are necessary.
- The objective function is concave and the constraint function is convex, so the Kuhn-Tucker conditions are sufficient.
- The Kuhn-Tucker conditions are
\[
3 - 2(x + 1)\lambda \leq 0, \text{ with equality if } x > 0
\]
\[
1 - 2(y + 1)\lambda \leq 0, \text{ with equality if } y > 0
\]
\[
(x + 1)^2 + (y + 1)^2 \leq 5 \text{ and } \lambda(5 - (x + 1)^2 - (y + 1)^2) = 0
\]
- The unique solution of these conditions is \((x, y, \lambda) = (1, 0, 3/4)\). Thus \((x, y) = (1, 0)\) is the solution of the problem.

-1.

- You write the Lagrangean either as
\[
M(x, k) = \sum_i p_i x_i - C(x_1, \ldots, x_n) - D(k) - \sum_i \lambda_i (x_i - k),
\]
in which case the Kuhn-Tucker conditions are
\[
p_i - C'_i(x_1, \ldots, x_n) - \lambda_i \leq 0, x_i \geq 0, \text{ and }
\]
\[
x_i(p_i - C'_i(x_1, \ldots, x_n) - \lambda_i) = 0 \text{ for } i = 1, \ldots, n
\]
\[
-D'(k) + \sum_i \lambda_i \leq 0, k \geq 0, \text{ and } k(-D'(k) + \sum_i \lambda_i) = 0
\]
\[
\lambda_i \geq 0, x_i \leq k, \text{ and } \lambda_i(x_i - k) = 0 \text{ for } i = 1, \ldots, n
\]
or as
\[
L(x, k) = \sum_i p_i x_i - C(x_1, \ldots, x_n) - D(k) - \sum_i \lambda_i (x_i - k) - \sum_i \mu_i x_i - \eta k,
\]
in which case the Kuhn-Tucker conditions are
\[
p_i - C'_i(x_1, \ldots, x_n) - \lambda_i - \mu = 0
\]
\[
-D'(k) + \sum_i \lambda_i - \eta = 0
\]
\[
\lambda_i \geq 0, \mu_i \geq 0, \eta \geq 0 \text{ for } i = 1, \ldots, n
\]
\[
\lambda_i(x_i - k) = 0, \mu_i x_i = 0, \text{ and } \eta k = 0 \text{ for } i = 1, \ldots, n
\]
- Each constraint function is concave, so the Kuhn-Tucker conditions are necessary. Each constraint function is convex and the objective function is concave (by the result of question 2), so the Kuhn-Tucker conditions are sufficient. That is, \((x^*, k^*)\) solves the problem if and only if there exist \(\lambda_i, \ldots, \lambda_n\) such that \(((x^*, k^*), \lambda_i, \ldots, \lambda_n)\) solves the Kuhn-Tucker conditions.
- If \(x_1 = x_2 = k > 0\) then from the Kuhn-Tucker conditions we have
\[ 1 - 2k - \lambda_i = 0 \]
\[ 3 - 2k - \lambda_c = 0 \]
\[ -2k + \lambda_i + \lambda_c = 0. \]

- From the third equation we have \(-2k = -\lambda_i - \lambda_c\), so the first two equations are

\[ 1 - 2\lambda_i - \lambda_c = 0 \]
\[ 3 - \lambda_i - 2\lambda_c = 0 \]

which imply that \(3\lambda_i = -1\), contradicting \(\lambda_i \geq 0\). Thus there is no solution in which \(x_i = x_c = k > 0\).

- If \(0 < x_i < k = x_c\) then from the Kuhn-Tucker conditions we have \(\lambda_i = 0\) and

\[ 1 - 2x_i = 0 \]
\[ 3 - 2k - \lambda_c = 0 \]
\[ -2k + \lambda_c = 0. \]

- From the first equation we have \(x_i = 1/2\). From the second and third equations we have \(k = 3/4\) and \(\lambda_c = 3/2\). Since \(\lambda_c \geq 0\), these values satisfy all the conditions. Thus a solution to the problem is \((x_i, x_c, k) = (1/2, 3/4, 3/4)\).

- Each constraint function is concave, so the KT conditions are necessary: every solution of the problem is a solution of the KT conditions. The conditions of the extreme value theorem are satisfied, so the problem has a solution.

Alternatively, the objective function is quasiconcave, the constraint functions are convex, and the gradient of the objective function is never \((0,0)\), so the solutions of the KT conditions are solutions of the problem.

The Lagrangean is

\[ M(x, y) = x^{\nu_2} + y - \lambda(px + y - I). \]

The Kuhn-Tucker conditions are

\[ (1/2)x^{\nu_2} - \lambda p \leq 0, \; x \geq 0 \quad \text{and} \quad x((1/2)x^{\nu_2} - \lambda p) = 0 \]
\[ 1 - \lambda \leq 0, \; y \geq 0 \quad \text{and} \quad y(1 - \lambda) = 0 \]
\[ \lambda \geq 0, \; px + y \leq I, \; \text{and} \; \lambda(px + y - I) = 0. \]

From the first condition, \(x > 0\), so that \(\lambda > 0\) and \(x = 1/(2\lambda p)^{1/\nu_2}\). Now either \(\lambda = 1\) or \(y = 0\):

- if \(\lambda = 1\) then \(x = 1/(4p^\nu_2), \; y = I - 1/(4p)\). This gives a solution if \(y \geq 0\), which implies that \(I \geq 1/(4p)\), or \(p \geq 1/(4I)\).
We conclude that the solution is
$$ \begin{cases} (I/p, 0) & \text{if } p \leq 1/(4I) \end{cases} $$

- The Kuhn-Tucker conditions are
  $$ f'(x^*) - \lambda^* p_i \leq 0, x_i^* \geq 0, \text{ and } x_i( f'(x^*) - \lambda^* p_i) = 0, \quad i = 1, 2 $$
  $$ p_x x_i^* + p_x y_i^* \leq c, \lambda^* \geq 0, \text{ and } \lambda^*(c - (p_x x_i^* + p_x y_i^*)) = 0 $$
- Or, alternatively,
  $$ f'(x^*) - \lambda^* p + \mu^*_i = 0, \quad i = 1, 2 $$
  $$ \mu^*_i \geq 0, x_i^* \geq 0, \text{ and } \mu^*_i x_i^* = 0 \text{ for } i = 1, 2 $$
  $$ p_x x_i^* + p_x y_i^* \leq c, \lambda^* \geq 0, \text{ and } \lambda^*(c - (p_x x_i^* + p_x y_i^*) - c) = 0 $$
- Each constraint function is concave (in fact, linear), so the Kuhn-Tucker conditions are necessary.
- The objective function is quasiconcave and each constraint function is quasiconvex, so if \( \nabla f(x^*) \neq (0,0) \) and \( (x_i^*, x_i^*) \) satisfies the Kuhn-Tucker conditions then \( x^* \) solves the problem.
- By the envelope theorem, \( F'(p_i, p_2, c) \) is the partial derivative of the Lagrangean with respect to \( p_i \) evaluated at \( (x_i^*, x_i^*, \lambda^*) \). Thus \( F'(p_i, p_2, c) = -\lambda^*(p_i, p_2, c)x_i^*(p_i, p_2, c) \).
- In this case the Kuhn-Tucker conditions are
  $$ x_i^* - \lambda p_i \leq 0, x_i \geq 0, \text{ and } x_i( x_i^* - \lambda p_i) = 0 $$
  $$ 2x_i - \lambda p_i \leq 0, x_i \geq 0, \text{ and } x_i(2x_i - \lambda p_i) = 0 $$
  $$ p_x x_i + p_x y_i \leq c, \lambda \geq 0, \text{ and } \lambda(p_x x_i + p_x y_i - c) = 0 $$
- If \( \lambda = 0 \) then from the first condition we have \( x_i = 0 \). But \( \nabla f(x_i, x_i) = (x_i^2, 2x_i) \), which is \( (0,0) \) for \( x_i = 0 \). Thus there is no solution of this type.
- If \( \lambda > 0 \) then \( p_x x_i + p_x y_i = c \). If \( x_i = 0 \) then \( \nabla f(x_i, x_i) = (0,0) \), as above. If \( x_i = 0 \) then \( x_i = 0 \) by the second set of conditions. Thus \( x_i > 0 \) and \( x_i > 0 \), so that by the first two conditions we have \( x_i^* = \lambda p_i \) and \( 2x_i^* = \lambda p_i \), so that \( x_i^2/p_i = 2x_i x_i/p_i \), or \( p_x x_i = 2p_x x_i \). Substituting into the constraint \( p_x x_i + p_x y_i = c \) we get \( x_i = c/(3p_i) \), \( x_i = 2c/(3p_i) \), and \( \lambda = 4c^2/(9p_i^2) \).
- Thus the solution of the problem is \((x_i, 7.3)\) Exercises on optimization problems with inequality constraints
1. For each possible value of the constant \(a\), solve the problem
\[
\max_{x, y} x + ay \text{ subject to } x^2 + y^2 \leq 1 \text{ and } x + y \geq 0.
\]

2. Consider the following problem.
\[
\max_{x_1, x_2} -x_1 - x_2 \text{ subject to } x_1 - 2x_2 \leq -1 \text{ and } 2x_1 + x_2 \leq 2
\]

a. Are the Kuhn-Tucker conditions necessary for a solution of this problem?
b. Are the Kuhn-Tucker conditions sufficient for a solution of this problem?
c. If possible, use the Kuhn-Tucker conditions to find the solution(s) of the problem.

[Solutions] 7.3 Solutions to exercises on optimization problems with inequality constraints

1. The Kuhn-Tucker conditions are necessary and sufficient because \(f\) is concave, each \(g_i\) is convex, and there exists a point \(x\) such that \(g_i(x) < c_i\) for each \(j\) (for example, \((1/4, 1/4))\).

The Kuhn-Tucker conditions are
\[
\begin{align*}
1 - 2\lambda_x x + \lambda_z &= 0 \\
-2\lambda_y y + \lambda_z &= 0 \\
\lambda_z &\geq 0, x^2 + y^2 \leq 1, \text{ and } \lambda_z (x^2 + y^2 - 1) = 0 \\
\lambda_z &\geq 0, x + y \geq 0, \text{ and } \lambda_z (x + y) = 0
\end{align*}
\]

From the first condition, we have \(\lambda_z > 0\) (otherwise \(\lambda_z = -1\)). Thus \(x^2 + y^2 = 1\). Now consider the two cases for the value of \(\lambda_z\).

If \(\lambda_z = 0\), we have \(x = 1/(2\lambda_z)\) and \(y = a/(2\lambda_z)\) from the first two conditions, so that \(\lambda_z = (1/2)\sqrt{(1 + a^2)}\) (using \(x^2 + y^2 = 1\)). Thus \((x, y) = (1/\sqrt{(1 + a^2)}, a/\sqrt{(1 + a^2)})\). For this pair to satisfy the constraint \(x + y \geq 0\) we need \(a \geq -1\).

If \(\lambda_z > 0\), we have \(x + y = 0\), so that from \(x^2 + y^2 = 1\) we have \((x, y) = (1/\sqrt{2}, -1/\sqrt{2})\) or \((x, y) = (-1/\sqrt{2}, 1/\sqrt{2})\). From the first two Kuhn-Tucker conditions we have \(\lambda_z = (1 - a)/(2(x - y))\). Thus for \((x, y) = (1/\sqrt{2}, -1/\sqrt{2})\) we have \(\lambda_z = (1 - a)\sqrt{2}/4\) and hence \(\lambda_z = -(1 + a)/2\), which are both nonnegative if and only if \(a \leq -1\). For \((x, y) = (-1/\sqrt{2}, 1/\sqrt{2})\) we have \(\lambda_z = -(1 - a)\sqrt{2}/4\) and hence \(\lambda_z = -(1 + a)/2\), which are not both nonnegative for any value of \(a\).

We conclude that if \(a \geq -1\) the solution of the problem is \((1/\sqrt{(1 + a^2)}, a/\sqrt{(1 + a^2)})\), with \(\lambda_z = (1/2)\sqrt{(1 + a^2)}\) and \(\lambda_z = 0\), and if \(a < -1\) the solution is \((1/\sqrt{2}, -1/\sqrt{2})\), with \(\lambda_z = (1 - a)\sqrt{2}/4\) and \(\lambda_z = -(1 + a)/2\).
2.

a. Yes: since each $g_j$ is concave.

b. Yes, since $f$ is concave and each $g_j$ is convex.

c. The Kuhn-Tucker conditions are

\[-2x_i - x_i - \lambda_i - 2\lambda_2 = 0 \]
\[-x_i - 2x_i + 2\lambda_i - \lambda_2 = 0\]
\[\lambda_i \geq 0, x_i - 2x_i \leq -1, \text{ and } \lambda_i(x_i - 2x_i + 1) = 0\]
\[\lambda_2 \geq 0, 2x_i + x_i \leq 2, \text{ and } \lambda_2(2x_i + x_i - 2) = 0\]

d. Consider the possible cases in turn:

e. $\lambda_i = \lambda_2 = 0$: From the first two equations we have $x_i = x_2 = 0$, which violates the first constraint.

f. $\lambda_i > 0$ and $\lambda_2 > 0$, so that $x_i - 2x_i = -1$ and $2x_i + x_i = 2$: We have $x_i = 3/5$ and $x_2 = 4/5$, which do not satisfy the first first-order condition.

g. $\lambda_i > 0$ and $\lambda_2 = 0$: Since $\lambda_i > 0$ we have $x_i = 2x_i - 1$; from the first-order conditions we have $x_i = -4/14$, $x_2 = 5/14$, and $\lambda_2 = 3/14$. Since $\lambda_i \geq 0$, a solution of the problem is $(x_i, x_2) = (-4/14, 5/14)$.

h. $\lambda_i = 0$ and $\lambda_2 > 0$: Since $\lambda_2 > 0$ we have $x_i = 2 - 2x_i$; from the first-order conditions we have $x_i = 1$, $x_2 = 0$, and $\lambda_2 = -1$. Since $\lambda_2 < 0$, this case does not yield a solution.

i. Thus unique solution to the problem is $(x_i, x_2) = (-4/14, 5/14)$. 
8.1 Differential equations: introduction

Ordinary differential equations

Many economic models with a temporal dimension involve relationships between the values of variables at a given point in time and the changes in these values over time. A model of economic growth, for example, typically contains a relationship between the change in the capital stock and the value of output. When time is modeled as a discrete variable (taking the values 1, 2, 3, ..., for example), the relationships may be modeled as difference equations, studied in the next section. When time is modeled as a continuous variable, they may be modeled as differential equations.

An ordinary differential equation takes the form

\[ G(t, x(t), x'(t), x''(t), ...) = 0 \]

for all \( t \), where \( t \) is a scalar variable (typically but not necessarily interpreted as "time"), \( G \) is a known function, \( x \) is an unknown function, \( x'(t) \) is the derivative of \( x \) with respect to \( t \), \( x''(t) \) is the second derivative of \( x \) with respect to \( t \), and so on. (In expositions of the theory of differential equations, derivatives are usually denoted by dots over the variable rather than primes beside the variable; I use primes because of the current limitations of HTML.) The word "ordinary" indicates that the unknown function \( x \) has only one argument; an equation involving the partial derivatives of a function of more than one variable is known as a "partial differential equation".

Ordinary differential equations may be classified according to the highest degree of derivative involved. If the highest degree is \( n \), the equation is called an \( n \)th order ordinary differential equation. A first-order ordinary differential equation, for example, takes the form \( G(t, x(t), x'(t)) = 0 \), and may alternatively be written as

\[ x'(t) = F(t, x(t)) \]

for all \( t \).

Similarly a second-order ordinary differential equation takes the form \( G(t, x(t), x'(t), x''(t)) = 0 \). Note that an \( n \)th order equation may or may not involve derivatives of degree less than \( n \).

Example

Here are some examples of ordinary differential equations:

\[ x'(t) - 1 = 0 \] for all \( t \) (first-order)

\[ x''(t) - 1 = 0 \] for all \( t \) (second-order)

\[ x''(t) - 2tx(t) = 0 \] for all \( t \) (second-order)

\[ x''(t)(x'(t))^{1/2} - t/x(t) = 0 \] for all \( t \) (second-order).

Solutions

A solution of an ordinary differential equation is a function \( x \) that satisfies the equation for all values of \( t \). A solution of the equation \( x'(t) - 1 = 0 \), for example, is the function \( x \) defined by \( x(t) = t \) for all \( t \). You will immediately notice that this equation has many other solutions: for any value of \( C \), the function \( x \) defined by \( x(t) = t + C \) for all \( t \) is a solution.
This multiplicity of solutions is normal: most differential equations with at least one solution have many solutions. A function that includes parameters (like $C$ in the previous example) is a "general solution" if every solution is equal to the function for some value of the parameters.

**Definition**

The function $f$ of the variable $t$ and a vector $C$ of parameters is a **general solution** of an ordinary differential equation if for every solution $x$ of the equation there is a value of $C$ such that $x(t) = f(t, C)$ for all $t$.

**Example**

The general solution of the differential equation $x'(t) - 1 = 0$ is the function $f$ defined by $f(t, C) = t + C$ for all $t$, where $C$ is a scalar.

A common though slightly informal way to express the conclusion of this example is that "the general solution of the equation is $x(t) = t + C$." That is, the symbol $C$ in a solution is taken to be a constant that may take any value. The same is true for the symbols $C_1$, $C_2$, and so on.

**Example**

Consider the differential equation $x''(t) - 1 = 0$. From the previous example we know that $x'(t) = t + C_1$, and hence the general solution of the equation is $x(t) = t^2/2 + C_1t + C_2$.

**Initial value problems**

Many models specify both that a function satisfy a differential equation and that the value of the function, or the values of the derivatives of the function, take certain values for some values of the variable. A model may specify, for example, that $x'(t) - 1 = 0$ and $x(0) = 1$, or that $x''(t) - 1 = 0$, $x'(1) = 4$, and $x(1) = 1$. The additional conditions on the values of the function are referred to as **initial conditions** (though these conditions may specify the value of $x$ or the value of its derivatives at any value of $t$, not necessarily the "first" value). A differential equation together with a set of initial conditions is called an **initial value problem**.

If the initial conditions are sufficiently restrictive, an initial value problem may have a unique solution, as the following examples illustrate.

**Example**

Consider the initial value problem $x'(t) - 1 = 0$ and $x(0) = 1$. The general solution of the differential equation is $x(t) = t + C$. Given $x(0) = 1$ we deduce that $C = 1$, so that the unique solution of the problem is $x(t) = t + 1$.

**Example**

Consider the initial value problem $x''(t) - 1 = 0$, $x'(1) = 4$, and $x(1) = 1$. The general solution of the differential equation is $x(t) = t^2/2 + C_1t + C_2$. We have $x'(t) = t + C_1$, so that $x'(1) = 1 + C_1$ and hence in any solution of the initial value problem we have $C_1 = 3$. We have also $x(1) = 1/2 + 3 + C_2$, so the condition $x(1) =$
1 implies that \( C_1 = -5/2 \). Thus the unique solution of the problem is \( x(t) = t/2 + 3t - 5/2 \).

These examples illustrate a general principle: an initial value problem in which the differential equation involves only the first derivative has a unique solution if it has one initial condition, and an initial value problem in which the differential equation involves only the first and second derivatives has a unique solution if it has two initial conditions.

**Equilibrium and stability**

If for some initial conditions a differential equation has a solution that is a constant function (independent of \( t \)), then the value of the constant is called an *equilibrium state* or *stationary state* of the differential equation. If, for all initial conditions, the solution of the differential equation converges to the equilibrium as \( t \) increases without bound, then the equilibrium is *globally stable*. If, for all initial conditions sufficiently close to the equilibrium the solution converges to the equilibrium as \( t \) increases without bound, then the equilibrium is *locally stable*. Otherwise the equilibrium is *unstable*.

**Example**

Consider the differential equation
\[
x'(t) + x(t) = 2.
\]
The general solution of this equation, as we shall see later, is
\[
x(t) = Ce^{-t} + 2.
\]
Thus for the initial condition \( x(0) = 2 \), the solution of the problem is \( x(t) = 2 \) for all \( t \). Thus the equilibrium state of the system is 2.

As \( t \) increases without bound, \( e^{-t} \) converges to zero, so the equilibrium is globally stable.

### 8.2 First-order differential equations: existence of a solution

A *first-order ordinary differential equation* may be written as
\[
x'(t) = F(t, x(t)) \text{ for all } t.
\]
Before trying to find a solution of such an equation, it is useful to know whether a solution exists.

We may think about the question of the existence of a solution with the help of a diagram known as a *direction field* or *integral field*. We plot \( t \) on the horizontal axis and \( x \) on the vertical axis, and for each of a set of pairs \((t, x)\) draw a short line segment through \((t, x)\) with slope equal to \( F(t, x(t)) \), which is equal to \( x'(t) \).

**Example**

The direction field of the equation
\[
x'(t) = x(t)t
\]
is shown in the figure below. For example, at every point \((t, 0)\) and every point
(0, x) we have \( x'(t) = 0 \), so the slope of the line segment through every such point is 0. Similarly, at every point (1, x) we have \( x'(t) = x \), so the slope of the line segment through (1, x) is \( x \) for each value of \( x \). (The grid size in the figure is 1/2.)

If the differential equation has a solution that passes through \((t, x)\), then the slope \( x'(t) \) of the solution at this point is equal to the slope of the line segment through \((t, x)\). Thus we can construct a rough solution starting from any initial condition by following the slopes of the line segments. The blue line in the following figure is a sketch of such a solution for the equation in the previous example.
This construction suggests that any initial value problem

\[ x'(t) = F(t, x(t)) \]
\[ x(t_0) = x_0 \]

in which the slopes of the line segments in the direction field change continuously as \((t, x)\) changes---that is, in which \(F\) is continuous---has a solution. If \(F\) is continuously differentiable, then in fact the initial value problem has a unique solution. Precisely, the following result may be shown.

**Proposition**

If \(F\) is a function of two variables that is continuous at \((t_0, x_0)\) then there exists a number \(a > 0\) and a continuously differentiable function \(x\) defined on the interval \((t_0 - a, t_0 + a)\) such that \(x(t_0) = x_0\) and \(x'(t) = F(t, x(t))\) for all \(t\) in the interval. If the partial derivative of \(F\) with respect to \(x\) is continuous on an open rectangle containing \((t_0, x_0)\) then there exists \(a > 0\) such that the initial value problem has a unique solution on the interval \((t_0 - a, t_0 + a)\).

The condition guaranteeing a unique solution (that the partial derivative of \(F\) with respect to \(x\) be continuous) is relatively mild, and is satisfied in almost all the examples we study. After looking at some direction fields, you might wonder how even an initial value problem that does not satisfy the condition could have more than one solution. Here is an example.

**Example**

Consider the initial value problem

\[ x'(t) = (x(t))^{1/2} \]
\[ x(0) = 0. \]

This problem does not satisfy the condition in the proposition for a unique solution, because the square root function is not differentiable at 0.

The problem has two solutions: \(x(t) = 0\) for all \(t\), and \(x(t) = (t/2)^2\) for all \(t\).

8.2 Exercises on first-order differential equations

1. Draw direction fields for the following differential equations.
   a. \(x'(t) = x/t\).
   b. \(x'(t) = -t/x\).
8.2 Solutions to exercises on first-order differential equations

1.
   a. The direction diagram for the first equation is given in the following figure.

   ![Direction Diagram](image)

   b. In the diagram for the second equation, the line segments point around concentric circles.

8.3 Separable first-order differential equations

A first-order ordinary differential equation that may be written in the form
\[ x'(t) = f(t)g(x) \]
for all \( t \)
is called **separable**.

**Example**
The equation
\[ x'(t) = \left[ e^{x(t)}/x(t) \right] \sqrt{1 + t^2} \]
is separable because we can write it as
\[ x'(t) = \left[ e^{x(t)}/x(t) \right] \cdot \left[ e^{\sqrt{1 + t^2}} \right]. \]

**Example**
The equation
\[ x'(t) = F(t) + G(x(t)) \]
is not separable unless either $F$ or $G$ is identically 0: it cannot be written in the form $x'(t) = f(t)g(x)$.

If the function $g$ is a constant, independent of $x$, then the general solution of the equation is simply the indefinite integral of $f$.

Even if $g$ is not constant, the equation may be easily solved. Assuming that $g(x) \neq 0$ for all values that $x$ assumes in a solution, we may write it as

$$\frac{dx}{g(x)} = f(t)dt.$$ 

Then we may integrate both sides, to get

$$\int (1/g(x))dx = \int f(t)dt,$$

where $\int h(y)dy$ is the indefinite integral of $h$ evaluated at $y$. Assuming we can perform the integrations, the resulting expression defines a solution.

If $g(x^*) = 0$ for some $x^*$ then $x(t) = x^*$ for all $t$ is also a solution.

Example

Consider the differential equation

$$x'(t) = x(t) + t,$$

for which we drew a direction field previously. First write the equation as

$$\frac{dx}{x} = t dt.$$ 

Then integrate both sides, to obtain

$$\ln x = \frac{t^2}{2} + C.$$ 

(I have consolidated the constants of integration in $C$ on the right-hand side.) Finally, isolate $x$ to obtain

$$x(t) = Ce^{t^2/2}$$

for all $t$.

The $C$ in this equation is equal to $e^C$ from the previous equation. I follow standard practice in using the same letter $C$ to denote the new constant.

This argument is valid only if $x(t) \neq 0$ for all $t$; thus in all the solutions it generates we need $C \neq 0$. However, looking at the original differential equation we see that the function $x$ defined by $x(t) = 0$ for all $t$ is also a solution.

If we have an initial condition $x(t_0) = x_0$ then the value of $C$ is determined by the equation

$$x_0 = Ce^{t_0^2/2}.$$ 

Example

Consider the differential equation

$$x'(t) = -2(x(t))t.$$
We may write this equation as

\[-dx/x^2 = 2t \, dt,\]

which may be integrated to yield

\[1/x = t + C,\]

yielding the set of solutions

\[x(t) = 1/(t + C) \text{ for all } t.\]

In addition, \(x(t) = 0\) for all \(t\) is a solution.

As before, an initial condition determines the exact solution. If \(x(0) = 1\), for example, we need

\[1 = 1/C\]

so \(C = 1\). Thus for this initial condition the solution is

\[x(t) = 1/(t + 1).\]

To take another example, if \(x(0) = 0\), then the solution is \(x(t) = 0\) for all \(t\).

Example

Consider the differential equation

\[x'(t) = t/((x(t))^a + 1).\]

Separating the variables yields

\[(x^a + 1)dx = t \, dt.\]

The value of \(x^a + 1\) is not 0 for any value of \(x\), so all the solutions of the equation are obtained by integrating both sides of this equation, which leads to

\[x^5/5 + x = t/2 + C.\]

Thus a solution \(x\) satisfies

\[(x(t))^5/5 + x(t) = t/2 + C \text{ for all } t.\]

In this case we cannot isolate \(x(t)\)---we cannot find the solutions explicitly, but know only that they satisfy this equation.

Example (Solow’s model of economic growth)

Output is produced from capital, denoted \(K\), and labor, denoted \(L\), according to the production function \(AK^{1-a}L^a\), where \(A\) is a positive constant and \(0 < a < 1\). A constant fraction \(s\) of output is "saved" (with \(0 < s < 1\)), and used to augment the capital stock. Thus the capital stock changes according to the differential equation

\[K'(t) = sA(K(t))^{1-a}L(t)^a\]

and takes the value \(K_0\) at \(t = 0\). The labor force is \(L_0 > 0\) at \(t = 0\) and grows at a
constant rate $\lambda$, so that

$$L'(t)/L(t) = \lambda.$$ 

One way of finding a solution of this model is to first solve for $L$, then substitute the resulting expression into the equation for $K'(t)$ and solve for $K$.

The equation for $L$ is separable, and may be written as

$$\frac{dL}{L} = \lambda \, dt.$$ 

The equation yields $\ln L = \lambda t + C$, or $L(t) = Ce^{\lambda t}$. Given the initial condition, we have $C = L_0$.

Substituting this result into the equation for $K'(t)$ yields

$$K'(t) = sA(K(t))^{1-a}(Le^{\lambda t})^a = sA(L_0)e^{a\lambda t}(K(t))^{1-a}.$$ 

This equation is separable, and may be written as

$$K^{-a}dK = sA(L_0)e^{a\lambda t}dt.$$ 

Integrating both sides, we obtain

$$K/a = sA(L_0)e^{a\lambda t}/a\lambda + C,$$ 

so that

$$K(t) = (sA(L_0)e^{a\lambda t}/a\lambda + C)^{1/a}.$$ 

Given $K(0) = K_0$, we conclude that $C = (K_0)^{1-a} - sA(L_0)/a\lambda$. Thus

$$K(t) = [sA(L_0)(e^{a\lambda t} - 1)/a\lambda + (K_0)^{1-a}]^{1/a}$$ 

for all $t$.

An economically interesting feature of the model is the evolution of the capital-labor ratio $K(t)/L(t)$. We have

$$K(t)/L(t) = \frac{[sA(L_0)(e^{a\lambda t} - 1)/a\lambda + (K_0)^{1-a}]^{1/a}}{(Le^{\lambda t})}$$ 

for all $t$.

As $t$ grows without bound, this fraction converges to

$$\frac{sA/\lambda}{a}.$$ 

A more general version of this model is studied in

1. **8.3 Exercises on separable first-order differential equations**

   a. $(x(t))^2x'(t) = t + 1$.
   b. $x'(t) = t - t$.
   c. $x'(t) = te^{-t}$.
   d. $e^{\omega x}x'(t) = t + 1$.
   e. $x'(t) = e^{-\omega t}/t x(t)$. 

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f. $x'(t) = 4tx(t) + t$.

2. Solve the following initial value problems.
   a. $tx'(t) = x(t)(1 - t)$, $(t, x) = (1, 1/e)$.
   b. $(1 + t)x'(t) = tx(t)$, $(t, x) = (0, 2)$.
   c. $x(t)x'(t) = t$, $(t, x) = (\sqrt{2}, 1)$.
   d. $e^{x(t)}(t - (x(t))^2 - 2x(t) - 1 = 0$, $(t, x) = (0, 0)$.
   e. $x'(t) = 4tx(t) + t$, $(t, x) = (0, 0)$.

8.3 Solutions to exercises on separable first-order differential equations

1.
   a. $x(t) = ((3/2)t^2 + 3t + 3C)^{1/3}$. (Separable.)
   b. $x(t) = t^4/4 - t^2/2 + C$. (Direct integration.)
   c. $x(t) = te^t - e^t - (1/2)t^2 + C$. (Direct integration.)
   d. $x(t) = \ln((1/2)t^2 + t + C)$. (Separable.)
   e. The equation is separable:
      \[
      \int xe^x \, dx = \int (1/t) \, dt,
      \]
      so (integrating by parts on the left) $xe^t - e^t = \ln t + C$. Thus the solution is defined by the condition
      $(x(t) - 1)e^{x(t)} = \ln t + C$.
   f. The equation is separable:
      \[
      \int (1/(4x + 1)) \, dx = \int t \, dt,
      \]
      so that
      $\frac{1}{4} \ln(4x + 1) = (1/2)t^2 + C$,
      or
      $x(t) = C \exp(2t) - 1/4$.

2.
   a. $x(t) = Cte^t; C = 1$.
   b. $x(t) = C(1 + t)^{1/3}; C = 2$.
   c. $x(t) = \sqrt{t + C}; C = -1$.
   d. $x(t) = (2 - C - e^t)/(C + e^t); C = 1$.
   e. $x(t) = Ce^{2t} - 1/4; C = 1/4$.

8.4 Linear first-order differential equations

General form
A linear first-order differential equation takes the form
\[ x'(t) + a(t)x(t) = b(t) \] for all $t$.
for some functions $a$ and $b$.

**Coefficient of $x(t)$ constant**

Consider the case in which $a(t) = a \neq 0$ for all $t$, so that $x'(t) + ax(t) = b(t)$ for all $t$.

If the left-hand side were the derivative of some function and we could find the integral of $b$ then we could solve the equation by integrating each side. Now, the left-hand side looks something like the derivative of a product. But for it to be *exactly* the derivative of a product of the form $f(t)x(t)$ we would need $f(t) = 1$ and $f'(t) = a$ for all $t$, which is clearly not possible.

Now suppose that we multiply both sides by $g(t)$ for each $t$. Then we have

$$g(t)x'(t) + ag(t)x(t) = g(t)b(t) \text{ for all } t.$$  

For the left-hand side of this equation to be the derivative of a product of the form $f(t)x(t)$ we need $f(t) = g(t)$ and $f'(t) = ag(t)$. Is there any function $f$ with this property? Yes! If $f(t) = e^w$ then $f'(t) = ae^w = a f(t)$.

Thus if we set $g(t) = e^w$, so that we have

$$e^w x'(t) + ae^w x(t) = e^w b(t),$$

then the integral of the left-hand side is $e^w x(t)$, and hence the solution of the equation is given by

$$e^w x(t) = C + \int e^w b(s) \, ds,$$

(where $\int f(s) \, ds$ is, as before, the indefinite integral of $f(s)$ evaluated at $t$), or

$$x(t) = e^{-w} [C + \int e^w b(s) \, ds].$$

This conclusion is summarized in the following result.

**Proposition**
The general solution of the differential equation

$$x'(t) + ax(t) = b(t) \text{ for all } t,$$

where $a$ is a constant and $b$ is a continuous function, is given by

$$x(t) = e^{-w} [C + \int e^w b(s) \, ds] \text{ for all } t.$$ 

Because multiplying the original equation by $e^w$ allows us to integrate the left-hand side, we call $e^w$ an **integrating factor**.

If $b(t) = b$ for all $t$ then the solution simplifies to

$$x(t) = Ce^w + bla.$$ 

Looking at the original equation we see that $x'(t) = 0$ if and only if $x(t) = bla$. Thus $x = bla$ is an equilibrium state.
For the initial condition \( x(t_0) = x_0 \), we have \( x_0 = Ce^{-at_0} + \frac{b}{a} \), so that \( C = (x_0 - \frac{b}{a})e^{at_0} \), and hence the solution is

\[
x(t) = (x_0 - \frac{b}{a})e^{-at} + \frac{b}{a}.
\]

Thus as \( t \) increases without bound, \( x(t) \) converges to \( \frac{b}{a} \) if \( a > 0 \), and grows without bound if \( a < 0 \) and \( x_0 \neq \frac{b}{a} \). That is, the equilibrium is globally stable if \( a > 0 \) and unstable if \( a < 0 \).

**Example**

Consider the differential equation

\[
x'(t) + 2x(t) = 6.
\]

The general solution of this equation is

\[
x(t) = Ce^{-2t} + 3.
\]

For the initial condition \( x(0) = 10 \), we obtain \( C = 7 \), so that the solution is

\[
x(t) = 7e^{-2t} + 3.
\]

This solution is stable, because \( 2 > 0 \).

**Example**

When the price of a good is \( p \), the total demand is \( D(p) = a - bp \) and the total supply is \( S(p) = \alpha + \beta p \), where \( a, b, \alpha, \) and \( \beta \) are positive constants. When demand exceeds supply, price rises, and when supply exceeds demand it falls. The speed at which the price changes is proportional to the difference between supply and demand. Specifically,

\[
p'(t) = \lambda[D(p) - S(p)]
\]

with \( \lambda > 0 \). Given the forms of the supply and demand functions, we thus have

\[
p'(t) + \lambda(b + \beta)p(t) = \lambda(a - \alpha).
\]

The general solution of this differential equation is

\[
p(t) = Ce^{-(b+\beta)t} + \frac{(a - \alpha)}{(b + \beta)}.
\]

The equilibrium price is \( (a - \alpha)/(b + \beta) \), and, given \( \lambda(b + \beta) > 0 \), this equilibrium is globally stable.

**Coefficient of \( x(t) \) dependent on \( t \)**

We may generalize these arguments to the differential equation

\[
x'(t) + ax(t) = b(t),
\]

in which \( a \) is a function. For \( g(t) \) times the left-hand side of this equation to be the derivative of \( f(t)x(t) \) we need \( f(t) = g(t) \) and \( f'(t) = a(t)g(t) \) for all \( t \). The function \( g(t) = e^{\int a(s)ds} \) has this property, because the derivative of \( \int a(s)ds \) is \( a(t) \) (by the fundamental theorem of calculus). Multiplying the equation by this factor, we have

\[
e^{\int a(t)dt}x'(t) + a(t)e^{\int a(s)ds}x(t) = e^{\int a(s)ds}b(t),
\]

or

\[
(d/dt)[x(t)e^{\int a(t)dt}] = e^{\int a(t)dt}b(t).
\]
Thus
\[ x(t)e^{\int_a^{\infty}b(u)du} = C + \int e^{\int_a^{\infty}b(u)du} \, du, \]
or
\[ x(t) = e^{\int_a^{\infty}b(u)du} [C + \int e^{\int_a^{\infty}b(u)du} \, du]. \]
In summary, we have the following result.

**Proposition**

The general solution of the differential equation
\[ x'(t) + a(t)x(t) = b(t) \]
for all \( t \), where \( a \) and \( b \) are continuous functions, is given by
\[ x(t) = e^{\int_a^{\infty}b(u)du} [C + \int e^{\int_a^{\infty}b(u)du} \, du] \]
for all \( t \).

**Example**

Consider the differential equation
\[ x'(t) + \frac{1}{t}x(t) = e^t. \]

We have
\[ \int \left( \frac{1}{s} \right) ds = \ln t \]
so
\[ e^{\int_1^{\infty} b(u)du} = t. \]

Thus the solution of the equation is
\[
\begin{align*}
x(t) &= \frac{1}{t}(C + \int ue^udu) \\
&= \frac{1}{t}(C + te^t - \int e^udu) \\
&= \frac{1}{t}(C + te^t - e^t) \\
&= C/t + e^t - e^t/t.
\end{align*}
\]

(I used integration by parts to obtain the second line.) We can check that this solution is correct by differentiating:
\[ x'(t) + x(t)/t = -C/t^2 + e^t - e^t/t + e^t/t + C/t + e/t - e/t^2 = e^t. \]

As before, an initial condition determines the value of \( C \).

### 8.4 Exercises on first-order linear differential equations

- Find the general solution of \( x'(t) + (1/2)x(t) = 1/4 \). Determine the equilibrium state and examine its stability.
- Find the general solutions of the following differential equations, and in each case solve the associated initial value problem for \( x(0) = 1 \).
- \( x'(t) - 3x(t) = 5. \)
- \( 3x'(t) + 2x(t) + 16 = 0. \)
- \( x'(t) + 2x(t) = t'. \)

- Solve the following differential equations.
  - \( tx'(t) + 2x(t) + t = 0 \) for \( t \neq 0. \)
  - \( x'(t) - x(t)/t = t \) for \( t > 0. \)
  - \( x'(t) - tx(t)/(t^2 - 1) = t \) for \( t > 1. \)
  - \( x'(t) - 2x(t)/t + 2a^2/t^2 = 0 \) for \( t > 0. \)

- In the theory of auctions, an initial value problem of the form
  \[ x'(t)G(t) + x(t)G'(t) = tG'(t) \]
  where \( G \) is a known function, arises. (The variable \( t \) is interpreted as a player's valuation of the object for sale in the auction.) Solve this equation, expressing the solution in terms of the function \( G \) (but not its derivative). (If you start by converting the equation into the standard form and finding the integrating factor, reflect afterwards on the fact that you did not need to do so.)

8.4 Solutions to exercises on first-order linear differential equations

- \( x(t) = Ce^{\alpha t} + 1/2. \) Equilibrium: \( x^* = 1/2; \) stable.
  - \( x(t) = Ce^{\alpha t} - 5/3; \) \( C = 8/3. \)
  - \( x(t) = Ce^{-2\alpha t} - 8; \) \( C = 9. \)
  - \( x = Ce^{\alpha t} + (1/2)t^2 - (1/2)t + 1/4 \) (integrate by parts twice); \( C = 3/4. \)

- Writing the equation in the standard form, we have
  \[ x'(t) + (2/t)x(t) = -1. \]
  Now, \( \int (2/s)ds = 2 \ln t, \) so from the general solution of an equation of the form \( x'(t) + a(t)x(t) = b(t) \) we have
  \[ x(t) = e^{2\ln t}[C - \int e^{2\ln u}du] \]
  or
  \[ x(t) = (1/t)(C - t\ln u) \]
  or
  \[ x(t) = (1/t)[C - \ln t] \]
  \[ = Ct - t/3. \]

- We have \( \int (-1/t)dt = -\ln t, \) so the integrating factor is \( 1/t. \) Hence the solution is \( x(t) = t(C + t). \)
  - \( x(t) = C(t-1)^{\alpha} + t - 1. \)
  - \( x(t) = Ct^2 + 2a^2/3t. \)

- The left-hand side of the differential equation is the derivative of \( x(t)G(t). \) Thus integrating both sides leads to
Applying integration by parts to the right-hand side produces
\[ t G(t) - \int t G(s) ds. \]
Thus the general solution of the differential equation is
\[ x(t) = t - \frac{\int t G(s) ds}{G(t)}. \]
The condition \( x(t_0) = t_0 \) implies that the solution of the initial value problem is
\[ x(t) = t - \frac{\int_{t_0}^t G(s) ds}{G(t)}. \]

**8.5 Differential equations: phase diagrams for autonomous equations**

We are often interested not in the exact form of the solution of a differential equation, but
only in the qualitative properties of this solution. In economics, in fact, the differential
equations that arise in our models usually contain functions whose forms we do not
specify explicitly, so there is no question of finding explicit solutions. One way of
studying the qualitative properties of the solutions of a differential equation is to
construct a **phase diagram**. I discuss this technique for the class of "autonomous" first-
order differential equations.

**Autonomous equations**

A first-order differential equation is **autonomous** if it takes the form \( x'(t) = F(x(t)) \) (i.e. the value of \( x'(t) \) does not depend independently on the variable \( t \)).

An **equilibrium state** of such an equation is a value of \( x \) for which \( F(x) = 0 \). (If \( F(x) = 0 \) then \( x'(t) = 0 \), so that the value of \( x \) does not change.)

A phase diagram indicates the sign of \( x'(t) \) for a representative collection of values of \( x \).
To construct such a diagram, plot the function \( F \), which gives the value of \( x' \). For values of \( x \) at which the graph of \( F \) is above the \( x \)-axis we have \( x'(t) > 0 \), so that \( x \) is increasing;
for values of \( x \) at which the graph is below the \( x \)-axis we have \( x'(t) < 0 \), so that \( x \) is decreasing. A value of \( x \) for which \( F(x) = 0 \) is an equilibrium state.

If \( x^* \) is an equilibrium and \( F'(x^*) < 0 \), then if \( x \) is slightly less than \( x^* \) it increases, whereas if \( x \) is slightly greater than \( x^* \) it decreases. If, on the other hand, \( F'(x^*) > 0 \), then if \( x \) is slightly less than \( x^* \) it decreases, moving further from the equilibrium, and if \( x \) is slightly greater than \( x^* \) it increases, again moving away from the equilibrium. That is, if \( x^* \) is an equilibrium then

7. if \( F'(x^*) < 0 \) then \( x^* \) is **locally stable**
8. if \( F'(x^*) > 0 \) then \( x^* \) is **unstable**.

The example in the following figure has three equilibrium states, \( a, b, \) and \( c \). The arrows on the \( x \)-axis indicate the direction in which \( x \) is changing (given by the sign of \( x'(t) \)) for each possible value of \( x \).
We see that the equilibrium $b$ is locally stable, whereas the equilibria $a$ and $c$ are unstable.

**Example (Solow's model of economic growth)**

The following model generalizes the one in an earlier example. Suppose that the production function is a strictly increasing and strictly concave function $F$ that is homogeneous of degree 1 (i.e. has "constant returns to scale"), rather than taking the specific form assumed in the earlier example.

We now have

$$K'(t) = sF(K(t), L(t)).$$

As before, the labor force grows at the constant rate $\lambda$, so that

$$L'(t)/L(t) = \lambda.$$

We may study the behavior of the capital-labor ratio $K(t)/L(t)$ as follows. Let $k = K/L$ and define the function $f$ of a single variable by

$$f(k) = F(k, 1)$$

for all $k$.

Since $F$ is increasing and concave we have $f' > 0$ and $f'' < 0$, and since it is homogeneous of degree 1 we have

$$F(K, L) = LF(K/L, 1) = Lf(k).$$

Thus

$$K'(t) = sL(t)f(k(t)).$$

Now,

$$k'(t) = [K'(t)L(t) - K(t)L'(t)]/(L(t))^2 = [K'(t) - k(t)L'(t)]/L(t)$$

so
\[ k'(t) = sf(k(t)) - \lambda k(t). \]

The phase diagram of this equation is shown in the following figure.

We see that the equation has two equilibria, one in which \( k^* = 0 \), which is unstable, and one in which \( sf(k^*) = \lambda k^* \), which is locally stable.

The stable equilibrium value of \( k \) depends on \( s \): the equation

\[ sf(k^*) = \lambda k^* \]

implicitly defines \( k^* \) as a function of \( s \). Which value of \( s \) maximizes the equilibrium value of per capita consumption? Per capita consumption is equal to \((1 - s)F(K, L)/L\), or, given our previous calculations, \((1 - s) f(k^*)\). Thus in an equilibrium per capita consumption is equal to \((1 - s) f(k^*(s))\). A stationary point of this function satisfies

\[ (1 - s) f'(k^*(s)) k^*(s) = f(k^*(s)). \]

Now, differentiating the equation defining \( k^* \) and rearranging the terms we obtain

\[ k^*(s) = f(k^*(s))/[\lambda - sf'(k^*(s))]. \]

Combining the last two equations we deduce that if \( s \) maximizes per capita consumption then

\[ f'(k^*(s)) = \lambda. \]

That is, the marginal product of capital is equal to the rate of population growth.

8.5 Exercises on qualitative theory of differential equations

6. Draw the phase diagrams associated with the following differential equations and determine the nature of the possible equilibrium states.
   1. \( x'(t) = x(t) - 1. \)
   2. \( x'(t) + 2x(t) = 24. \)
   3. \( x'(t) = x(t)^2 - 9. \)
4. $x'(t) = x(t)^3 + x(t)^2 - x(t) - 1.$
5. $x'(t) = 3x(t)^3 + 1.$
6. $x'(t) = x(t)e^t.$
7. $x'(t) = (x(t) + 2)(x(t))^2.$

8.5 Solutions to exercises on qualitative theory of differential equations

13. 

1. The phase diagram is shown below. $x = 1$ is unstable.

![Phase diagram 1](image1)

2. The phase diagram is shown below. $x = 12$ is stable.

![Phase diagram 2](image2)

3. The phase diagram is shown below. $x = -3$ is stable; $x = 3$ is unstable.

![Phase diagram 3](image3)
4. Note that $x'(t) = (x(t) - 1)(x(t) + 1)$. $x = 1$ is unstable; $x = -1$ is unstable. (It is stable from the right and unstable from the left.) The phase diagram is shown below.

5. The phase diagram is shown below. There are no equilibrium states.

6. The phase diagram is shown below. There is a single equilibrium, $x = 0$, which is unstable.
7. The phase diagram is shown below. There are two equilibria, \( x = -2 \) and \( x = 0 \). The equilibrium \( x = -2 \) is unstable; the equilibrium \( x = 0 \) is unstable. (It is stable from the left, but not from the right.)

8.6 Second-order differential equations

**General form**

A **second-order ordinary differential equation** is a differential equation of the form

\[ G(t, x(t), x'(t), x''(t)) = 0 \]

for all \( t \), involving only \( t \), \( x(t) \), and the first and second derivatives of \( x \). We can write such an equation in the form

\[ x''(t) = F(t, x(t), x'(t)). \]

**Equations of the form** \( x''(t) = F(t, x'(t)) \)

An equation of the form

\[ x''(t) = F(t, x'(t)), \]

in which \( x(t) \) does not appear, can be reduced to a first-order equation by making the substitution \( z(t) = x'(t) \).

**Example**

Let \( u(w) \) be a utility function for wealth \( w \). The function

\[ \rho(w) = -wu''(w)/u'(w) \]

is known as the **Arrow-Pratt measure of relative risk aversion**. (If \( \rho_u(w) > \rho_v(w) \) for two utility functions \( u \) and \( v \) then \( u \) reflects a greater degree of risk-aversion
than does $v$.)

What utility functions have a degree of risk-aversion that is independent of the level of wealth? That is, for what utility functions $u$ do we have

$$a = -wu''(w)/u'(w) \text{ for all } w?$$

This is a second-order differential equation in which the term $u(w)$ does not appear. (The variable is $w$, rather than $t$.) Define $z(w) = u'(w)$. Then we have

$$a = -wz'(w)/z(w)$$

or

$$az(w) = -wz'(w),$$

a separable equation that we can write as

$$a \cdot dw/w = -dz/z.$$ 

The solution is given by

$$a \cdot \ln w = -\ln z(w) + C,$$

or

$$z(w) = Cw^{-a}.$$ 

Now, $z(w) = u'(w)$, so to get $u$ we need to integrate:

$$u(w) = C \ln w + B \text{ if } a = 1$$

We conclude that a utility function with a constant degree of risk-aversion equal to $a$ takes this form.

**Linear second-order equations with constant coefficients**

A **linear second-order differential equation with constant coefficients** takes the form

$$x''(t) + ax'(t) + bx(t) = f(t)$$

for constants $a$ and $b$ and a function $f$. Such an equation is **homogeneous** if $f(t) = 0$ for all $t$.

Let $x_1$ be a solution of the equation
\[ x''(t) + ax'(t) + bx(t) = f(t), \]
to which I refer subsequently as the "original equation". For any other solution of this equation \( x \), define \( z = x - x_1 \). Then \( z \) is a solution of the homogeneous equation

\[ x''(t) + ax'(t) + bx(t) = 0 \]
(because \( z''(t) + az'(t) + bz(t) = [x''(t) + ax'(t) + bx(t)] - [x_1''(t) + ax_1'(t) + bx_1(t)] = f(t) - f(t) = 0 \)). Further, for every solution \( z \) of the homogeneous equation, \( x_1 + z \) is clearly a solution of original equation.

We conclude that the set of all solutions of the original equation may be found by finding one solution of this equation and adding to it the general solution of the homogeneous equation. That is, we may use the following procedure to solve the original equation.

**Procedure for finding general solution of linear second-order differential equation with constant coefficients**

The general solution of the differential equation

\[ x''(t) + ax'(t) + bx(t) = f(t) \]

may be found as follows.

1. **Find the general solution of the associated homogeneous equation** \( x''(t) + ax'(t) + bx(t) = 0 \).
2. **Find a single solution of the original equation** \( x''(t) + ax'(t) + bx(t) = f(t) \).
3. **Add together the solutions found in steps 1 and 2.**

I now explain how to perform to the first two steps and illustrate step 3.

**1. Finding the general solution of a homogeneous equation**

You might guess, based on the solutions we found for first-order equations, that the homogeneous equation has a solution of the form \( x(t) = Ae^r \). Let's see if it does. If \( x(t) = Ae^r \), then \( x'(t) = rAe^r \) and \( x''(t) = r^2Ae^r \), so that

\[ x''(t) + ax'(t) + bx(t) = r^2Ae^r + arAe^r + bAe^r = Ae^r(r^2 + ar + b). \]

Thus for \( x(t) \) to be a solution of the equation we need

\[ r^2 + ar + b = 0. \]

This equation is known as the **characteristic equation** of the differential equation. It has either two distinct real roots, a single real root, or two complex roots. If \( a^2 > 4b \), then it has two distinct real roots, say \( r \) and \( s \), and we have shown that both \( x(t) = Ae^r \) and \( x(t) = Be^s \), for any values of \( A \) and \( B \), are solutions of the equation. Thus also \( x(t) = Ae^r + Be^s \) is a solution. It can be shown, in fact, that every solution of the equation takes this form.

The cases in which the characteristic equation has a single real root \( (a^2 = 4b) \) or complex roots \((a^2 < 4b) \) require a slightly different analysis, with the following conclusions.

**Proposition**
Consider the homogeneous linear second-order differential equation with constant coefficients
\[ x''(t) + ax'(t) + bx(t) = 0 \]
for all \( t \), where \( a \) and \( b \) are numbers. The general solution of this equation depends on the character of the roots of the characteristic equation \( r^2 + ar + b = 0 \) as follows.

**Distinct real roots**
If \( a^2 > 4b \), in which case the characteristic equation has distinct real roots, the general solution of the equation is
\[ Ae^r + Be^{-r}. \]

**Repeated real root**
If \( a^2 = 4b \), in which case the characteristic equation has a single root, the general solution of the equation is
\[ (A + Br)e^r, \]
where \( r = -(1/2)a \) is the root.

**Complex roots**
If \( a^2 < 4b \), in which case the characteristic equation has complex roots, the general solution of the equation is
\[ (A\cos(\beta t) + B\sin(\beta t))e^{\alpha t}, \]
where \( \alpha = -a/2 \) and \( \beta = \sqrt{(b - a^2/4)} \). This solution may alternatively be expressed as
\[ Ce^{\alpha t}\cos(\beta t + \omega), \]
where the relationships between the constants \( C, \omega, A, \) and \( B \) are \( A = C \cos \omega \) and \( B = -C \sin \omega \).

**Example**
Consider the differential equation
\[ x''(t) + x'(t) - 2x(t) = 0. \]
The characteristic equation is
\[ r^2 + r - 2 = 0 \]
so the roots are 1 and -2. That is, the roots are real and distinct. Thus the general solution of the differential equation is
\[ x(t) = Ae^t + Be^{-2t}. \]

**Example**
Consider the differential equation
\[ x''(t) + 6x'(t) + 9x(t) = 0. \]
The characteristic equation has a repeated real root, equal to -3. Thus the general solution of the differential equation is
\[ x(t) = (A + Bt)e^{-3t}. \]
Consider the equation
\[ x''(t) + 2x'(t) + 17x(t) = 0. \]

The characteristic roots are complex. We have \( a = 2 \) and \( b = 17 \), so \( \alpha = -1 \) and \( \beta = 4 \), so the general solution of the differential equation is
\[ [A \cos(4t) + B \sin(4t)]e^{-t}. \]

2. Finding a solution of a nonhomogeneous equation

A good technique to use to find a solution of a nonhomogeneous equation is to try a linear combination of \( f(t) \) and its first and second derivatives. If, for example, \( f(t) = 3t - 6t^2 \), then try to find values of \( A, B, \) and \( C \) such that \( A + Bt + Ct^2 \) is a solution. Or if \( f(t) = 2\sin t + \cos t \), then try to find values of \( A \) and \( B \) such that \( f(t) = Asin t + Bcos t \) is a solution. Or if \( f(t) = 2e^{at} \) for some value of \( B \), then try to find a value of \( A \) such that \( Ae^{at} \) is a solution.

**Example**

Consider the differential equation
\[ x''(t) + x'(t) - 2x(t) = t. \]

The function on the right-hand side is a second-degree polynomial, so to find a solution of the equation, try a general second-degree polynomial---that is, a function of the form \( x(t) = C + Dt + Et^2 \). (The important point to note is that you should not restrict to a function of the form \( x(t) = Et^2 \).) For this function to be a solution,
\[ 2E + D + 2Et - 2C - 2Dt - 2Et^2 = t \]
for all \( t \),

so we need (equating the coefficients of \( t^2 \), \( t \), and the constant on both sides),

\[
\begin{align*}
2E + D - 2C &= 0 \\
2E - D &= 0 \\
-2E &= 1.
\end{align*}
\]

We deduce that \( E = -1/2, \ D = -1/2, \) and \( C = -3/4 \). That is,
\[ x(t) = -3/4 - t/2 - t/2 \]
is a solution of the differential equation.

3. Add together the solutions found in steps 1 and 2

This step is quite trivial!

**Example**

Consider the equation from the previous example, namely
\[ x''(t) + x'(t) - 2x(t) = f. \]

We saw above that the general solution of the associated homogeneous equation is

\[ x(t) = Ae^t + Be^{-2t}. \]

and that

\[ x(t) = -3/4 - t/2 - t^2/2 \]

is a solution of the original equation.

Thus the general solution of the original equation is

\[ x(t) = Ae^t + Be^{-2t} - 3/4 - t/2 - t^2/2. \]

**Stability of solutions of homogeneous equation**

Consider the homogeneous equation

\[ x''(t) + ax'(t) + bx(t) = 0. \]

If \( b \neq 0 \), this equation has a single equilibrium, namely 0. (That is, the only constant function that is a solution is equal to 0 for all \( t \).) To consider the stability of this equilibrium, consider separately the three possible forms of the general solution of the equation.

**Characteristic equation has two real roots**

- If the characteristic equation has two real roots, \( r \) and \( s \), so that the general solution of the equation is \( Ae^r + Be^s \), then the equilibrium is stable if and only if \( r < 0 \) and \( s < 0 \).

**Characteristic equation has a single real root**

- If the characteristic equation has a single real root, then the equilibrium is stable if and only if this root is negative. (Note that if \( r < 0 \) then for any value of \( k \), \( te^k \) converges to 0 as \( t \) increases without bound.)

**Characteristic equation has complex roots**

- If the characteristic equation has complex roots, the form of the solution of the equation is \( Ae^{\alpha} \cos(\beta t + \omega) \), where \( \alpha = -a/2 \), the real part of each root. Thus the equilibrium is stable if and only if the real part of each root is negative.

The real part of a real root is simply the root, so we can combine the three cases: the equilibrium is stable if and only if the real parts of both roots of the characteristic equation are negative. A bit of algebra shows that this condition is equivalent to \( a > 0 \) and \( b > 0 \).

If \( b = 0 \), then every number is an equilibrium, and none of these equilibria is stable.

In summary, we have the following result.
Proposition
An equilibrium of the homogeneous linear second-order differential equation \( x''(t) + ax'(t) + bx(t) = 0 \) is stable if and only if the real parts of both roots of the characteristic equation \( r^2 + ar + b = 0 \) are negative, or, equivalently, if and only if \( a > 0 \) and \( b > 0 \).

Example
Consider the following macroeconomic model. Denote by \( Q \) aggregate supply, \( p \) the price level, and \( \pi \) the expected rate of inflation. Assume that aggregate demand is a linear function of \( p \) and \( \pi \), equal to \( a - bp + c\pi \) where \( a > 0 \), \( b > 0 \), and \( c > 0 \). An equilibrium condition is \( Q(t) = a - bp(t) + c\pi(t) \).

Denote by \( Q^* \) the long-run sustainable level of output, and assume that prices adjust according to the equation
\[
p'(t) = h(Q(t) - Q^*) + \pi(t),
\]
where \( h > 0 \). Finally, suppose that expectations are adaptive:
\[
\pi'(t) = k(p'(t) - \pi(t))
\]
for some \( k > 0 \). Is this system stable?

One way to answer this question is to reduce the system to a single second-order differential equation by differentiating the equation for \( p'(t) \) to obtain \( p''(t) \) and then substituting in for \( \pi'(t) \) and \( \pi(t) \). We obtain
\[
p''(t) - h(ck - b)p'(t) + khbp(t) = kh(a - Q^*).
\]

We conclude the system is stable if and only if \( kc < b \). (Since \( k > 0 \), \( h > 0 \), and \( b > 0 \), we have \( kb > 0 \).)

In particular, if \( c = 0 \) (i.e. expectations are ignored) then the system is stable. If expectations are taken into account, however, and respond rapidly to changes in the rate of inflation (\( k \) is large), then the system may be unstable.

8.6 Exercises on second-order differential equations

3. Solve the following differential equations in general, and subject to the given initial conditions. In each case, determine if the solution is stable or unstable.
   a. \( x''(t) + 3x'(t) - 4x(t) = 12 \). Initial conditions: \( x(0) = 4, x'(0) = 2 \).
   b. \( x''(t) - 2x'(t) + x(t) = 3 \). Initial conditions: \( x(0) = 4, x'(0) = 2 \).
   c. \( x''(t) + 4x'(t) + 8x(t) = 2 \). Initial conditions: \( x(0) = 9/4, x'(0) = 4 \).
4. Find a particular solution of the differential equation

\[ x''(t) + 2x'(t) + x(t) = t. \]

5. Solve the differential equation \( x''(t) - 4x'(t) + 4x(t) = 5 \) in general, and subject to the initial conditions \( x(0) = 4 \) and \( x'(0) = 6 \).

### 8.6 Solutions to exercises on second-order differential equations

- The roots of the characteristic equation are 1 and \(-4\). A particular integral is \( x(t) = -3 \). Thus the general solution is
  
  \[ Ae^t + Be^{-4t} - 3. \]

  For the given initial conditions we have \( A = 6 \) and \( B = 1 \). The general solution is unstable.

- \( x(t) = A_1 e^t + A_2 e^{-4t} + 3; A_1 = 1 \) and \( A_2 = 1 \).
- \( x(t) = e^{-t}(A_{1}\cos 2t + A_{2}\sin 2t) + 1/4; A_1 = 2 \) and \( A_2 = 4 \).
- \( x(t) = t - 2 \).
- \( x(t) = A_{1,2}e^t + 5/4; A_1 = 11/4 \) and \( A_2 = 1/2 \). (Initial conditions: \( x(0) = 4 \), so that \( A_1 + 5/4 = 4 \), or \( A_1 = 11/4 \); \( x'(0) = 6 \), so that \( A_2 + 2A_1 = 6 \), or \( A_2 = 6 - 22/4 = 1/2 \).)

### 8.7 Systems of first-order linear differential equations

#### Two equations in two variables

Consider the system of linear differential equations (with constant coefficients)

\[
\begin{align*}
x'(t) & = ax(t) + by(t) \\
y'(t) & = cx(t) + dy(t).
\end{align*}
\]

We can solve this system using the techniques from the previous section, as follows. First isolate \( y(t) \) in the first equation, to give

\[ y(t) = x'(t)/b - ax(t)/b. \]

Now differentiate this equation, to give

\[ y'(t) = x''(t)/b - ax'(t)/b. \]

Why is this step helpful? Because we can now substitute for \( y(t) \) and \( y'(t) \) in the second of the two equations in our system to yield

\[ x''(t)/b - ax'(t)/b = cx(t) + d[x'(t)/b - ax(t)/b], \]

which we can write as

\[ x''(t) - (a + d)x'(t) + (ad - bc)x(t) = 0, \]

an equation we know how to solve!
Having solved this linear second-order differential equation in \( x(t) \), we can go back to the expression for \( y(t) \) in terms of \( x'(t) \) and \( x(t) \) to obtain a solution for \( y(t) \).

(We could alternatively have started by isolating \( x(t) \) in the second equation and creating a second-order equation in \( y(t) \).)

**Example**

Consider the system of equations

\[
\begin{align*}
 x'(t) &= 2x(t) + y(t) \\
 y'(t) &= -4x(t) - 3y(t).
\end{align*}
\]

Isolating \( y(t) \) in the first equation we have \( y(t) = x'(t) - 2x(t) \), so that \( y'(t) = x''(t) - 2x'(t) \). Substituting these expressions into the second equation we get

\[
x''(t) - 2x'(t) = -4x(t) - 3x'(t) + 6x(t),
\]

or

\[
x''(t) + x'(t) - 2x(t) = 0.
\]

**We have seen** that the general solution of this equation is

\[
x(t) = Ae^t + Be^{-2t}.
\]

Using the expression \( y(t) = x'(t) - 2x(t) \) we get

\[
y(t) = Ae^t - 2Be^{-2t} - 2Ae^t - 2Be^{-2t},
\]

or

\[
y(t) = -Ae^t - 4Be^{-2t}.
\]

**General linear systems**

We may write the system

\[
\begin{align*}
 x'(t) &= ax(t) + by(t) \\
 y'(t) &= cx(t) + dy(t),
\end{align*}
\]

studied above, as

\[
\begin{bmatrix}
 x'(t) \\
 y'(t)
\end{bmatrix} =
\begin{bmatrix}
 a & b \\
 c & d
\end{bmatrix}
\begin{bmatrix}
 x(t) \\
 y(t)
\end{bmatrix}.
\]

More generally, we may write a system of \( n \) simultaneous homogeneous linear equations in the \( n \) variables \( x_i(t) \) for \( i = 1, \ldots, n \) as
\( x'(t) = Ax(t), \)
where \( A \) is an \( n \times n \) matrix.

Now, if \( n = 1 \), in which case \( A \) is simply a number, we know that for the initial condition \( x(0) = C \) the equation has a unique solution

\[ x(t) = Ce^{\alpha}. \]

Here's a stunning result:

for every value of \( n \), the unique solution of the homogeneous system of linear differential equations \( x'(t) = Ax(t) \) subject to the initial condition \( x(0) = C \) is

\[ x(t) = Ce^{\alpha}. \]

You will immediately ask: what is \( e^\alpha \) when \( A \) is a matrix? Recall that if \( a \) is a number we have

\[ e^a = 1 + a/1! + a^2/2! + ..., \]

or, more precisely,

\[ e^a = \sum_{k=0}^{\infty} (a^k/k!). \]

Now, when \( A \) is a matrix we may define

\[ e^A = \sum_{k=0}^{\infty} (A^k/k!), \]

where \( A^0 \) is the identity matrix (and 0! = 1). (You know how to multiply matrices together, so you know how to compute the right hand side of this equation.) That's it! You can now find the solution of any homogeneous system of linear differential equations ... assuming that you can compute the infinite sum in the definition of \( e^{\alpha A} \). Therein lies the difficulty. There are techniques for finding \( e^{\alpha A} \), but they involve methods more advanced than the ones in this tutorial.

I give only one example, which shows how the trigonometric functions may emerge in the solution of a system of two simultaneous linear equations, which, as we saw above, is equivalent to a second-order equation.

**Example**

Consider the system

\[
\begin{pmatrix}
  x_1'(t) \\
  x_2'(t)
\end{pmatrix} =
\begin{pmatrix}
  0 & -b \\
  b & 0
\end{pmatrix}
\begin{pmatrix}
  x_1(t) \\
  x_2(t)
\end{pmatrix}.
\]

We need to find \( A^k \) for each value of \( k \), where

\[
A =
\begin{pmatrix}
  0 & -b \\
  b & 0
\end{pmatrix}.
\]

You should be able to convince yourself (by computing \( A^k, A^l, \) and \( A^i \)) that if \( k \) is odd we have

\[
A^k =
\begin{pmatrix}
  0 & (-1)^{(k+1)/2}b^k
\end{pmatrix}.
\]
whereas if $k$ is even we have
\[
A_k = \begin{pmatrix} (-1)^{k/2} b^k & 0 \\ 0 & 1 \end{pmatrix}.
\]

Given these results, we have
\[
e^A = \begin{pmatrix} 1 - b/2! + b^3/4! - b^5/6! + ... & -b + b^3/3! - b^5/5! + ... \\ -b + b^3/3! - b^5/5! + ... & 1 - b^2/2! + b^4/4! - b^6/6! + ... \end{pmatrix}.
\]

Now, you should recall that
\[
\sin b = b - b^3/3! + b^5/5! - ...
\]
and
\[
\cos b = 1 - b^2/2! + b^4/4! - b^6/6! + ...
\]
Thus
\[
e^A = \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}.
\]

We conclude that the solution of the system of equations given the initial conditions $x_1(0) = C_1$ and $x_2(0) = C_2$ is
\[
\begin{align*}
x_1(t) &= C_1 \cos bt - C_2 \sin bt \\
x_2(t) &= C_1 \sin bt + C_2 \cos bt.
\end{align*}
\]

You are asked, in an exercise, to verify this solution by using the technique discussed at the start of this section to convert the two-equation system to a single second-order linear differential equation.

You may wonder whether the emergence of the trigonometric functions in this example is accidental. It is not! Exponentials, the trigonometric functions, and complex numbers (which arise as roots of the characteristic equation in the technique of the previous section), are closely related. Specifically, for any value of $x$ we have
\[
e^x = \cos x + i \sin x,
\]
(where $i$ is the square root of $-1$). But that is another story.

(An excellent, but advanced, exposition of the material in this section is contained in Differential equations, dynamical systems, and linear algebra by Morris W. Hirsch and Stephen Smale.)

### 8.7 Exercises on systems of first-order linear differential equations

7. Find the general solution of the system of equations
\[
\begin{align*}
x'(t) &= 4y(t) \\
y'(t) &= -x(t) + 4y(t).
\end{align*}
\]
8. Find the general solution of the following system of equations and the particular solution with \( x(0) = 1 \) and \( y(0) = 0 \).

\[
\begin{align*}
x'(t) &= x(t) - 5y(t) \\
y'(t) &= 2x(t) - 5y(t).
\end{align*}
\]

9. Find the solution of the system

\[
\begin{align*}
x'(t) &= -by(t) \\
y'(t) &= bx(t)
\end{align*}
\]

with the initial conditions \( x(0) = C_1 \) and \( y(0) = C_2 \).

8.7 Solutions to exercises on systems of first-order linear differential equations

- Isolating \( y(t) \) in the first equation we have \( y(t) = x'(t)/4 \), so that \( y'(t) = x''(t)/4 \).

Substituting these expressions into the second equation we get

\[
x''(t)/4 = -x(t) + x'(t),
\]

or

\[
x''(t) - 4x'(t) + 4x(t) = 0.
\]

The characteristic equation is \( (r - 2)^2 \), which has the repeated root \( r = 2 \). Thus the general solution of this equation is

\[
x(t) = (A + Bt)e^{2t}.
\]

Given \( y(t) = x'(t)/4 \), we thus have

\[
y(t) = (1/2)(A + Bt)e^{2t} + (1/4)Be^{2t} = [(1/4)B + (1/2)A + (1/2)Bt]e^{2t}.
\]

- The two equations generate the second-order equation

\[
x''(t) + 4x'(t) + 5x(t) = 0.
\]

The characteristic equation has complex roots. The solution of the second-order equation is

\[
x(t) = e^{\omega t}(A \cos \omega t + B \sin \omega t).
\]

Calculating \( x' \) and substituting into the first equation we obtain

\[
y(t) = (1/5)e^{\omega}(3A - B)\cos \omega t + (3A - B)\sin \omega t.
\]

For the given initial conditions we have \( A = 1 \) and \( B = 3 \).

- We have \( x''(t) = -by'(t) = -b^2x(t) \), or

\[
x''(t) + b^2x(t) = 0.
\]

The characteristic equation is \( r^2 + b^2 = 0 \), which has complex roots. Thus, given the general form of the solution to such an equation, the solution of our equation is
\[ x(t) = C_1 \cos bt - C_2 \sin bt. \]

Using \( y(t) = -x'(t)/b \), we have \( y(t) = C_1 \sin bt + C_2 \cos bt \). (Remember that the derivative of \( \sin bt \) is \( b \cos bt \), and the derivative of \( \cos bt \) is \( -b \sin bt \).

### 9.1 First-order difference equations

A general **first-order difference equation** takes the form
\[ x_i = f(t, x_{i-1}) \]
for all \( t \).

We can solve such an equation by successive calculation: given \( x_0 \) we have
\[
\begin{align*}
x_1 &= f(1, x_0) \\
x_2 &= f(2, x_1) = f(2, f(1, x_0)) \\
& \quad \text{and so on.}
\end{align*}
\]

In particular, given any value \( x_0 \), there exists a unique solution path \( x_1, x_2, \ldots \).

However, calculating the solution in this way doesn't tell us much about the properties of the solution. We should like to have a general formula for the solution. If the form of \( f \) is simple, such formulas exist.

**First-order linear difference equations with constant coefficient**

A **first-order difference equation with constant coefficient** takes the form
\[ x_i = ax_{i-1} + b_i, \]
where \( b_i \) for \( t = 1, \ldots \) are constants. (Note that the constant that multiplies \( x_{i-1} \) is constant, while \( b_i \) may depend on \( t \).)

If we use the method of successive calculation above, we see a pattern:

\[ x_i = ax_0 + \sum_{k=1}^i a^{i-k}b_k. \]

We know that the equation has a unique solution path, by the argument above. Thus to check that this formula gives the unique solution it is enough to verify that it satisfies the original equation. We have
\[
\begin{align*}
ax_i + b_i &= a(ax_{i-1} + \sum_{k=1}^{i-1}a^{i-k}b_k) + b_i \\
&= ax_i + \sum_{k=1}^{i-1}a^{i-k}b_k + b_i \\
&= ax_i + \sum_{k=1}^{i}a^{i-k}b_k \\
&= x_i,
\end{align*}
\]

verifying that the solution is correct.

**Proposition**

For any given value \( x_0 \), the unique solution of the difference equation
\[ x_i = ax_{i-1} + b_i, \]

is
\[ x_i = ax_0 + \sum_{k=1}^i a^{i-k}b_k. \]

In the special case that \( b_i = b \) for all \( k = 1, \ldots \) we have
\[ x_t = a x_0 + b \sum_{i=0}^{t-1} a^i. \]

Now, the sum of any finite geometric series \( 1 + a + a^2 + \ldots + a^{t-1} \) is given by
\[ 1 + a + a^2 + \ldots + a^{t-1} = \frac{(1-a^t)}{(1-a)}. \]

if \( a \neq 1 \). Thus we have
\[ x_t = a x_0 + b \left( \frac{1}{1-a} \right). \]

if \( a \neq 1 \).

**Proposition**

For any given value \( x_0 \), the unique solution of the difference equation
\[ x_t = a x_{t-1} + b, \]

where \( a \neq 1 \), is
\[ x_t = a (x_0 - \frac{b}{1-a}) + \frac{b}{1-a}. \]

**Equilibrium**

In general, given the starting point \( x_0 \), the value of \( x \) changes with \( t \). Is there any value of \( x_0 \) for which \( x \) doesn’t change? Yes: clearly if
\[ x^* = \frac{b}{1-a} \]

then \( x_t \) is constant, equal to \( \frac{b}{1-a} \).

We call \( x^* \) the **equilibrium** value of \( x \). We can rewrite the solution as
\[ x_t = a (x_0 - x^*) + x^*. \]

**Qualitative behavior**

The qualitative behavior of the solution path depends on the value of \( a \).
\[ |a| < 1 \]

- \( x \) converges to \( x^* \): the solution is **stable**. There are two subcases:
  - \( 0 < a < 1 \)
    - Monotonic convergence.
  - \(-1 < a < 0 \)
    - Damped oscillations.

\[ |a| > 1 \]

- Divergence:
  - \( a > 1 \)
    - Explosion.
  - \( a < -1 \)
    - Explosive oscillations.

**Example**

Suppose that demand depends upon current price:
\[ D_t = \gamma - \delta p_t, \]
while supply depends on previous price:

\[ S_t = (p_{t-1} - \alpha)/2\beta. \]

(Maybe the lag exists because production takes time (think of agricultural production).) Note that these equations cannot be valid for all values of \( p_t \) and \( p_{t-1} \): \( D_t \) and \( S_t \) must both be nonnegative. The following arguments assume that \( p_t \) stays in the range for which the equations are valid.

For equilibrium in period \( t \) we need \( D_t = S_t \), which yields the difference equation

\[ p_t = -(1/2\beta\delta)p_{t-1} + (\alpha + 2\beta\gamma)/2\beta\delta. \]

The equilibrium price is given by

\[ p^* = (\alpha + 2\beta\gamma)/(1 + 2\beta\delta), \]

so that we can write the solution as

\[ p_t = p^* + (-1/2\beta\delta)(p_0 - p^*). \]

The equilibrium is stable if

\[ 2\beta\delta > 1. \]

A solution path for such parameters is shown in the following figure.

(If the system is unstable then eventually a boundary is reached, and the equations determining \( D_t \) and \( S_t \) above are no longer valid.)

In the previous example \( b_t \) is independent of \( t \). In the next two examples \( b_t \) depends on \( t \).
Example

You have assets of $z_0$. You can earn a constant return $r$ on these assets. (That is, after $t$ years you will have $(1 + r)z_t$ if you do not consume any of the assets.) The rate of inflation is $i$, where $i < r$. In each period $t \geq 1$ you withdraw an amount of money equivalent in purchasing power to $y$ in period 1. (That is, you withdraw $y(1 + i)^t$ in each period $t$.)

2. How long will your assets last if $r = 0.08$, $i = 0.04$, $y = 50,000$, and $z_0 = 1,000,000$?

3. How large do you assets need to be to last for 30 years of retirement if you want to withdraw $80,000$ per year during retirement (with $r = 0.08$ and $i = 0.04$)?

We have

$$z_t = (1 + r)z_{t-1} - (1 + i)^t y,$$

so

$$z_t = (1 + r)z_t - \sum_{s=1}^{t} (1 + r)^s (1 + i)^s y$$

$$= (1 + r)z_0 - y(1 + r)^t [1 - ((1 + i)/(1 + r))]/[1 - (1 + i)/(1 + r)]$$

$$= (1 + r)z_0 - y(1 + r) [1 - ((1 + i)/(1 + r))]/(r - i)$$

$$= (1 + r)[z_0 - y(1 - ((1 + i)/(1 + r)))/(r - i)].$$

Thus your assets last until the period $t$ for which

$$z_0 = (y/(r-i))(1 - ((1 + i)/(1 + r))).$$

Hence

$$t = \frac{\log(1 - (r-i)z_0/y)}{\log((1 + i)/(1 + r))}.$$

Thus in the example we have $t = 42.6$ years. The amount of money you need to last 30 years at $80,000$ per year is $z_0 = 1,355,340$. (If you look at the path of your total assets, you find they first increase, then gradually decrease.)

**First-order linear difference equations with variable coefficient**

The solution of the equation

$$x_t = ax_{t-1} + b,$$

can be found, as before, by successive calculation. We obtain

$$x_t = (\prod_{s=1}^{t} a_s)x_0 + \sum_{s=1}^{t} (\prod_{s=1}^{t} a_s) b_s,$$

where a product with no terms (e.g. from $t+1$ to $t$) is 1.
(There is an example in the book of compound interest in the case of a variable interest rate.)

9.1 Exercises on first-order difference equations

3. Solve the following difference equations. In each case, determine whether the solution path is convergent or divergent.
   - \( x_t = -3x_{t-1} + 4 \)
   - \( x_t = (1/2)x_{t-1} + 3 \)

4. Consider the model similar to the cobweb model described in class (and in Example 20.4 in the book) with the exception that the suppliers' price expectations are adaptive, so that

\[
P_t^* = P_{t-1}^* + \eta(p_t - P_t^*)
\]

where \( 0 < \eta \leq 1 \) and \( P_t^* \) is the expected price in period \( t \), and supply depends on the expected price:

\[
S_t = -\beta + \alpha P_t^*
\]

(Note that the parameters \( \alpha \) and \( \beta \) here are different from the parameters \( \alpha \) and \( \beta \) in the model presented in class.) The other components of the model remain \( D_t = \gamma - \delta p_t \) and \( S_t = D_t \).

   - Find the difference equation in \( p_t \) that this model generates (first isolate \( P_t^* \) in the supply function.)
   - Solve this difference equation for \( P_t \). For what values of \( \eta \) is the time path of \( p_t \) oscillatory? convergent?

5. Consider the difference equation

\[
y_{t+1}(a + by_t) = cy_t
\]

where \( a, b, \) and \( c \) are positive constants and \( y_0 > 0 \).

   - Show that \( y_t > 0 \) for all \( t \).
   - Define \( x_t = 1/y_t \). Show that by using this substitution, the new difference equation is of the type \( x_{t+1} = \alpha x_t + \beta \). Use this fact to solve the difference equation

\[
y_{t+1}(2 + 3y_t) = 4y_t
\]

with \( y_0 = 1/2 \). What is the limit of \( y_t \) at \( t \to \infty \)

9.1 Solutions to exercises on first-order difference equations

   - \( x_t = 1 + (-3)(x_0 - 1) \); divergent (and oscillating).
   - \( x_t = 6 + (1/2)(x_0 - 6) \); convergent.
After isolating $P^*$, in the supply function, substitute the expression you get into the equation for $P^*$, and then substitute for $S$, the expression for $D$, to get $p = (1 - \eta - \eta \alpha / \delta)p_{t+1} + \eta(\gamma + \beta)/\delta$.

$p = (p_0 - (\gamma + \beta)/(\delta + \alpha))(1 - \eta - \eta \alpha / \delta) + (\gamma + \beta)/(\delta + \alpha)$. Oscillates if $\eta > \delta/(\delta + \alpha)$; converges if $\eta < 2\delta/(\delta + \alpha)$.

Given that $a > 0$, $b > 0$, and $c > 0$ we have $y_0 > 0$ so long as $y_{t-1} > 0$. Since $y_0 > 0$ we thus have $y_t > 0$ for all $t$.

Substituting gives $x_{t+1} = (a/c)x_t + b/c$. Thus the solution of the specific equation given is, in the $x_t$ variable, $x_t = (1/2)^{t+1} + 3/2$, or $y_t = [(1/2)^{t+1} + 3/2]$. As $t \to \infty$ we have $y_t \to 2/3$.

9.2 Second-order difference equations

A general second-order difference equation takes the form

$$x_{t+2} + ax_{t+1} + bx_t = c_t.$$  

As for a first-order equation, a second-order equation has a unique solution: by successive calculation we can see that given $x_0$ and $x_1$ there exists a uniquely determined value of $x_t$ for all $t \geq 2$. (Note that for a second-order equation we need two starting values, $x_0$ and $x_1$, rather than one.)

Second order linear equations

Consider the equation

$$x_{t+2} + ax_{t+1} + bx_t = c_t.$$  

We solve this equation by first considering the associated homogeneous equation

$$x_{t+2} + ax_{t+1} + bx_t = 0.$$  

Suppose that we find two solutions of this equation, $u_t$ and $v_t$. Then

$$u_{t+2} + au_{t+1} + bu_t = 0$$  

and  

$$v_{t+2} + av_{t+1} + bv_t = 0.$$  

Hence for any constants $A$ and $B$ we have

$$(Au_{t+2} + Aau_{t+1} + Abu_t) + (Bv_{t+2} + Bav_{t+1} + Bbv_t) = 0,$$  

or

$$(Au_{t+2} + Bv_{t+2}) + (aAu_{t+1} + aBv_{t+1}) + (bAu_t + bBv_t) = 0.$$  

That is, $Au + Bv$ is also a solution of the equation.

This solution has two arbitrary constants in it, so it seems it might be a general solution. But in order to be a general solution, it must be that the two solutions are really independent; for example, they can't be proportional to each other. The solutions are linearly independent if

$$\begin{vmatrix} u_0 & v_0 \\ u_1 & v_1 \end{vmatrix} \neq 0.$$  

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If this condition is satisfied, then \( Au + Bv \) is a general solution of the homogeneous equation.

Now return to the original equation. Suppose that \( u^* \) is a solution of this equation. Let \( x \) be an arbitrary solution of the original equation. Then \( x - u^* \) is a solution of the homogeneous equation. Thus \( x - u^* = Au + Bv \) for some values of \( A \) and \( B \), or

\[
x = Au + Bv + u^*.
\]

That is, any solution of the original equation is the sum of a some solution of this equation and a solution of the homogeneous equation.

**Example**

Consider the equation

\[
x_{t+2} - 5x_{t+1} + 6x_t = 2t - 3.
\]

The associated homogeneous equation is

\[
x_{t+2} - 5x_{t+1} + 6x_t = 0.
\]

Two solutions of this equation are \( u_t = 2t \) and \( v_t = 3t \). (Shortly we'll see how to find such solutions.) These are linearly independent:

\[
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}
= 1.
\]

To find a solution of the original equation we can guess that it takes the form \( u^* = at + b \). In order for \( u^* \) to be a solution we need

\[
a(t+2) + b - 5[a(t+1) + b] + 6(at + b) = 2t - 3.
\]

Equating coefficients, we have \( a = 1 \) and \( b = 0 \). Thus \( u^* = t \) is a solution.

We conclude that the general solution of the equation is

\[
x_t = A2^t + B3^t + t.
\]

**Second-order linear equations with constant coefficients**

A **second-order linear equation with constant coefficients** takes the form

\[
x_{t+2} + ax_{t+1} + bx_t = c.
\]

The strategy for solving such an equation is very similar to the strategy for solving a **second-order linear differential equations with constant coefficients**. We first consider the associated **homogeneous equation**

\[
x_{t+2} + ax_{t+1} + bx_t = 0.
\]
The homogeneous equation

We need to find two solutions of the homogeneous equation
\[ x_{n+2} + ax_{n+1} + bx_n = 0. \]

We can guess that a solution takes the form \( u_t = m^t \). In order for \( u_t \) to be a solution, we need
\[
m(m^t + am + b) = 0
\]
or, if \( m \neq 0 \),
\[
m^t + am + b = 0.
\]
This is the characteristic equation of the difference equation. Its solutions are
\[-(1/2)a \pm \sqrt{(1/4)a^2 - b}.\]

We can distinguish three cases:

Distinct real roots
If \( a^2 > 4b \), the characteristic equation has distinct real roots, and the general solution of the homogeneous equation is
\[ Am_1^t + Bm_2^t, \]
where \( m_1 \) and \( m_2 \) are the two roots.

Repeated real root
If \( a^2 = 4b \), then the characteristic equation has a single root, and the general solution of the homogeneous equation is
\[ (A + Bt)m^t, \]
where \( m = -(1/2)a \) is the root.

Complex roots
If \( a^2 < 4b \), then the characteristic equation has complex roots, and the general solution of the homogeneous equation is
\[ Ar^t \cos(\theta t + \omega), \]
where \( A \) and \( \omega \) are constants, \( r = \sqrt{b} \), and \( \cos \theta = -a/(2\sqrt{b}) \), or, alternatively,
\[ Cr^t \cos(\theta t) + Cr^t \sin(\theta t), \]
where \( C_1 = A \cos \omega \) and \( C_2 = -A \sin \omega \) (using the formula that \( \cos(x+y) = (\cos x)(\cos y) - (\sin x)(\sin y) \)).

In each case the solutions are linearly independent:

3. In the first case the solutions \( u_t = m_1^t \) and \( v_t = m_2^t \) are linearly independent:
\[
\begin{vmatrix}
  u_0 & v_0 \\
  u_1 & v_1 \\
\end{vmatrix} = \begin{vmatrix} 1 & 1 \\
  m_1 & m_2 \end{vmatrix} = m_2 - m_1 \neq 0
\]

4. In the second case the solutions \( u_t = m^t \) and \( v_t = tm^t \) are linearly independent:
\[
\begin{vmatrix} 1 & 0 \end{vmatrix} = m \neq 0
\]
5. In the third case the solutions $u = r \cos(\theta t)$ and $v = r \sin(\theta t)$ are linearly independent:

$$
\begin{vmatrix}
1 & 0 \\
r \cos \theta & r \sin \theta \\
\theta & \theta
\end{vmatrix} = r \sin \theta = \sqrt{b\sqrt{(1 - \cos^2 \theta)} = \sqrt{b\sqrt{(1 - a^2/4b)} = \sqrt{(b - (1/4)a^2)} > 0}.
$$

In the third case, when the characteristic equation has complex root, the solution oscillates. $Ar$ is the amplitude (which depends on the initial conditions) at time $t$, and $r$ is growth factor. $\theta/2\pi$ is the frequency of the oscillations and $\omega$ is the phase (which depends on the initial conditions).

If $|r| < 1$ then the oscillations are damped; if $|r| > 1$ then they are explosive.

**Example**

Consider the equation

$$x_{n+2} + x_{n+1} - 2x_n = 0.$$ 

The roots of the characteristic equation are 1 and $-2$ (real and distinct). Thus the solution is

$$x_n = A + B(-2)^n.$$ 

**Example**

Consider the equation

$$x_{n+2} + 6x_{n+1} + 9x_n = 0.$$ 

The roots of the characteristic equation are $-3$ (real and repeated). Thus the solution is

$$x_n = (A + Bt)(-3)^n.$$ 

**Example**

Consider the equation

$$x_{n+2} - x_{n+1} + x_n = 0.$$ 

The roots of the characteristic equation are complex. We have $r = 1$ and $\cos \theta = 1/2$, so $\theta = (1/3)\pi$. So the general solution is

$$x_n = A\cos((1/3)\pi t + \omega).$$

The frequency is $(\pi/3)/2\pi = 1/6$ and the growth factor is 1, so the oscillations are undamped.
The original (nonhomogeneous) equation

To find the general solution of the original equation
\[ x_{,2} + ax_{,1} + bx = c, \]
we need to find one of its solutions. Suppose that \( b \neq 0 \).

The form of a solution depends on \( c \).

Suppose that \( c = c \) for all \( t \). Then \( x = C \) is a solution if \( C = c/(1 + a + b) \) and if \( 1 + a + b \neq 0 \). (If \( 1 + a + b = 0 \) then try \( x = C t \); if that doesn't work try \( x = C t^2 \).)

More generally, if \( c \) is a linear combination of terms of the form \( q t^r \), \( e^{pt} \), \( \cos(pt) \), and \( \sin(pt) \) (for some constants \( q \), \( p \), and \( m \)), and products of such terms, then the method of undetermined coefficients can be used, with a trial solution of the form suggested by \( c \), as illustrated in the following examples. If \( c \) happens to satisfy the homogeneous equation---and hence is part of the general solution already---then a different approach must be taken, which I do not discuss.

Example
Consider the equation
\[ x_{,2} - 5x_{,1} + 6x = 4t^2 + t + 3. \]

The general solution of the associated homogeneous equation is \( A 2^t + B 3^t \) (found by examining the characteristic equation, as before).

To find a particular solution, try
\[ x = C 4t + D t^2 + E t + F. \]

Substituting this potential solution into the equation and equating coefficients we find that
\[ C = 1/2, \quad D = 1/2, \quad E = 3/2, \quad \text{and} \quad F = 4 \]
gives a solution.

Thus the general solution of the equation is
\[ x = A 2^t + B 3^t + (1/2) 4t + (1/2)t^2 + (3/2)t + 4. \]
Stability
As for differential equations, we say that a system is stable if its long-run behavior is not sensitive to the initial conditions.

Consider the second-order equation
\[ x_{n+2} + ax_{n+1} + bx_n = c. \]
Write the general solution as
\[ x_n = Au_n + Bv_n + u^*, \]
where \( A \) and \( B \) are determined by the initial conditions. This solution is **globally asymptotically stable** (or simply **stable**) if the first two terms approach 0 as \( t \to \infty \), for all values of \( A \) and \( B \). In this case, for any initial conditions, the solution of the equation approaches the particular solution \( u^* \).

If the first two terms approach zero for all \( A \) and \( B \), then \( u_n \) and \( v_n \) must approach zero. (To see that \( u_n \) must approach zero, take \( A = 1 \) and \( B = 0 \); to see that \( v_n \) must approach 0, take \( A = 0 \) and \( B = 1 \).) A necessary and sufficient condition for this to be so is that the moduli of the roots of the characteristic equation be both less than 1. (The modulus of a complex number \( \alpha + \beta i \) is \( \sqrt{\alpha^2 + \beta^2} \), which is the absolute value of number if the number is real.)

There are two cases:

5. If the characteristic equation has complex roots then the modulus of each root is \( \sqrt{\beta} \) (the roots are \( \alpha \pm \beta i \), where \( \alpha = -a/2 \) and \( \beta = \sqrt{b - (1/4)a^2}) \). So for stability need \( b < 1 \).

6. If the characteristic equation has real roots then the modulus of each root is its absolute value. So for stability we need the absolute values of each root to be less than 1, or \( \left| -a/2 + \sqrt{(a^2/4 - \beta)} \right| < 1 \) and \( \left| -a/2 - \sqrt{(a^2/4 - \beta)} \right| < 1 \).

We can show (see the argument in the book on page 756, and problem 10 on page 758) that these conditions are equivalent to the conditions \( \left| a \right| < 1 + b \) and \( b < 1 \). Thus in summary we have:

**The solution is stable if and only if the modulus of each root of characteristic equation is less than 1. In terms of the coefficients of the terms in the equation, the solution is stable if and only if \( \left| a \right| < 1 + b \) and \( b < 1 \).**

### 9.2 Exercises on second-order difference equations

- Solve the following difference equations and determine whether the solution paths are convergent or divergent, oscillating or not.
  - \( x_{n+2} + 3x_{n+1} - (7/4)x_n = 9 \).
\[ \begin{align*}
& o \quad x_{n+2} - 2x_{n+1} + 2x = 1. \\
& o \quad x_{n+2} - x_{n+1} + (1/4)x = 2. \\
& o \quad x_{n+2} + 2x_{n+1} + x = 9.2'. \\
& o \quad x_{n+2} - 3x_{n+1} + 2x = 3.5' + \sin((1/2)\pi t)
\end{align*} \]

9.2 Solutions to exercises on second-order difference equations

4.

a. \( A_1(1/2) + A_2(-(7/2)) + 4 \). Nonconvergent oscillations.
b. \( \sqrt{2}(A_1 \cos(\pi/4)t + A_2 \sin(\pi/4)t) + 1 \). Nonconvergent oscillation.
c. \( A_1(1/2) + A_2t(1/2) + 8 \). Convergent, non-oscillating.
d. The characteristic equation is \( m^2 + 2m + 1 = (m+1)^2 = 0 \), which has a double root of \(-1\). So the general solution of the homogeneous equation is \( x = (C_1 + C_2t)(-1) \). A particular solution is obtained by inserting \( u^* = A_2t \), which yields \( A = 1 \). So the general solution of the inhomogeneous equation is \( x = (C_1 + C_2t)(-1) + 2 \).
e. By using the method of undetermined coefficients the constants \( A, B, \) and \( C \) in the particular solution \( u^* = A_5 + B\cos(\pi/2)t + C\sin(\pi/2)t \), we obtain \( A = 1/4, B = 3/10, \) and \( C = 1/10 \). So the general solution to the equation is

\[ x = C_1 + C_2t + (1/4)5 + (3/10)\cos((\pi/2)t) + (1/10)\sin((\pi/2)t). \]